On a Relative Computability Notion for Real Functions

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Abstract

- For any class $\mathcal{F}$ of total functions in $\mathbb{N}$, we define what it means for a real function to be conditionally $\mathcal{F}$-computable. This notion extends the notion of uniform $\mathcal{F}$-computability of real functions introduced in the paper [SkWeGe 10].
- If $\mathcal{F}$ consists of recursive functions then the conditionally $\mathcal{F}$-computable real functions are computable in the sense of [Gr 55] extended by allowing the used computable functionals to be partial and by considering real functions of any number of variables.
- Under certain weak assumptions about $\mathcal{F}$, we show that:
  - conditional $\mathcal{F}$-computability is preserved by substitution,
  - all conditionally $\mathcal{F}$-computable real functions are locally uniformly $\mathcal{F}$-computable,
  - the conditionally $\mathcal{F}$-computable real functions with compact domains are uniformly $\mathcal{F}$-computable.
- All elementary functions of calculus are conditionally $\mathcal{M}^2$-computable.

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Computability of Real Functions in the Extended Sense of [Gr 55]

As in [SkWeGe 10], a triple (f, g, h) of total one-argument functions in $\mathbb{N}$ will be called to name a real number ξ if

$$\left| \frac{f(t) - g(t)}{h(t) + 1} - \xi \right| < \frac{1}{t + 1}$$

for all $t \in \mathbb{N}$ ($h = \lambda t.t$ is actually used in [Gr 55]).

Let $N \in \mathbb{N}$ and $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$. The function θ is computable in the extended sense of [Gr 55] iff there exist recursive operators F, G, H acting on $3N$-tuples of one-argument functions in $\mathbb{N}$ and such that, whenever $(\xi_1, \ldots, \xi_N) \in D$ and $(f_1, g_1, h_1), \ldots,$ (f_N, g_N, h_N) are triples naming $\xi_1, \ldots, \xi_N$, respectively, the functions $F(\overline{f}, \overline{g}, \overline{h}), G(\overline{f}, \overline{g}, \overline{h}), H(\overline{f}, \overline{g}, \overline{h})$, where $\overline{f} = f_1, \ldots, f_N$, $\overline{g} = g_1, \ldots, g_N$, and $\overline{h} = h_1, \ldots, h_N$, are total, and the triple of them names $\theta(\xi_1, \ldots, \xi_N)$.

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Subrecursive Computability of Functions of Reals as a Certain Kind of Relative Computability

As far as we know, the first paper in this direction is [TeZi 10] (especially if its preliminary version at arxiv.org is taken into consideration). Further ones are [SkWeGe 10] and [Sk xx], where the uniform $\mathcal{F}$-computability is introduced and studied.

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$\mathcal{F}$-Substitutional Mappings

For any $m \in \mathbb{N}$, we will denote by $\mathbb{T}_m$ the set of all m-argument total functions in $\mathbb{N}$. Let $\mathcal{F} \subseteq \bigcup_{m \in \mathbb{N}} \mathbb{T}_m$. For any $k, m \in \mathbb{N}$, certain mappings of $\mathbb{T}_1^k$ into $\mathbb{T}_m$ will be called *$\mathcal{F}$-substitutional*, as follows:

1. For any m-argument projection function h in $\mathbb{N}$ the mapping F defined by $F(f_1, \ldots, f_k) = h$ is $\mathcal{F}$-substitutional.

2. For any $i \in \{1, \ldots, k\}$, if F_0 is a $\mathcal{F}$-substitutional mapping of $\mathbb{T}_1^k$ into $\mathbb{T}_m$ then so is the mapping F defined by

$$F(f_1, \ldots, f_k)(n_1, \ldots, n_m) = f_i(F_0(f_1, \ldots, f_k)(n_1, \ldots, n_m)).$$

3. For any $r \in \mathbb{N}$ and $f \in \mathcal{F} \cap \mathbb{T}_r$, if $F_1, \ldots, F_r$ are $\mathcal{F}$-substitutional mappings of $\mathbb{T}_1^k$ into $\mathbb{T}_m$ then so is the mapping F defined by

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Two Statements about $\mathcal{F}$-Substitutional Mappings

Proposition

Let $F : \mathbb{T}_1^k \to \mathbb{T}_m$ *and* $G_1, \ldots, G_m : \mathbb{T}_1^k \to \mathbb{T}_l$ *be* $\mathcal{F}$*-substitutional. Then so is the mapping* $H : \mathbb{T}_1^k \to \mathbb{T}_l$ *defined by*

$$H(\overline{f})(\overline{n}) = F(\overline{f})(G_1(\overline{f})(\overline{n}), \ldots, G_m(\overline{f})(\overline{n}))\,,$$

where $\overline{f} = f_1, \ldots, f_k$, *and* $\overline{n} = n_1, \ldots, n_l$.

Proposition

Let $F : \mathbb{T}_1^k \to \mathbb{T}_m$ *and* $G_1, \ldots, G_k : \mathbb{T}_1^l \to \mathbb{T}_{p+1}$ *be* $\mathcal{F}$*-substitutional. Then so is the mapping* $H : \mathbb{T}_1^l \to \mathbb{T}_{p+m}$ *defined by the equality*

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where $\overline{g} = g_1, \ldots, g_l$, $\overline{u} = u_1, \ldots, u_p$, *and* $\overline{n} = n_1, \ldots, n_m$.

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Conditional $\mathcal{F}$-Computability of Real Functions

Let $N \in \mathbb{N}$ and $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$. The function θ will be called *conditionally $\mathcal{F}$-computable* if there exist $\mathcal{F}$-substitutional mappings $E : \mathbb{T}_1^{3N} \to \mathbb{T}_1$ and $F, G, H : \mathbb{T}_1^{3N} \to \mathbb{T}_2$ such that, whenever $(\xi_1, \ldots, \xi_N) \in D$ and $(f_1, g_1, h_1), \ldots, (f_N, g_N, h_N)$ are triples from $\mathbb{T}_1^3$ naming $\xi_1, \ldots, \xi_N$, respectively, the following holds, where $\overline{f} = f_1, \ldots, f_N$, $\overline{g} = g_1, \ldots, g_N$, and $\overline{h} = h_1, \ldots, h_N$:

1. There exists a natural number s such that $E(\overline{f}, \overline{g}, \overline{h})(s) = 0$.
2. For any natural number s with $E(\overline{f}, \overline{g}, \overline{h})(s) = 0$, the number $\theta(\xi_1, \ldots, \xi_N)$ is named by the triple

$$(\lambda t.F(\overline{f}, \overline{g}, \overline{h})(s, t), \lambda t.G(\overline{f}, \overline{g}, \overline{h})(s, t), \lambda t.H(\overline{f}, \overline{g}, \overline{h})(s, t)).$$

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The Uniformly $\mathcal{F}$-Computable Real Functions are Conditionally $\mathcal{F}$-Computable

Let $N \in \mathbb{N}$, and let the function $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$, be uniformly $\mathcal{F}$-computable. Then there exist $\mathcal{F}$-substitutional mappings $F^\circ, G^\circ, H^\circ : \mathbb{T}_1^{3N} \to \mathbb{T}_1$ such that, whenever $(\xi_1, \ldots, \xi_N) \in D$ and $(f_1, g_1, h_1), \ldots, (f_N, g_N, h_N)$ are triples from $\mathbb{T}_1^3$ naming $\xi_1, \ldots, \xi_N$, respectively, the number $\theta(\xi_1, \ldots, \xi_N)$ is named by the triple $(F^\circ(\overline{f}, \overline{g}, \overline{h}), G^\circ(\overline{f}, \overline{g}, \overline{h}), H^\circ(\overline{f}, \overline{g}, \overline{h}))$, where $\overline{f} = f_1, \ldots, f_N$, $\overline{g} = g_1, \ldots, g_N$, and $\overline{h} = h_1, \ldots, h_N$.

To show the conditional $\mathcal{F}$-computability of θ, we set

$$\begin{aligned}
E(\overline{f}, \overline{g}, \overline{h})(s) &= s, \\
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\end{aligned}
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The Function $\lambda\xi.1/\xi$ is Conditionally $\mathcal{M}^2$-Computable

To prove this, we may set

$$\begin{aligned}
E(f,g,h)(s) &= (2h(s)+3) \dot{-} (s+1)|f(s)-g(s)|,\\
F(f,g,h)(s,t) &= (h(u(s,t))+1)\,\mathrm{sg}(f(u(s,t)) \dot{-} g(u(s,t))),\\
G(f,g,h)(s,t) &= (h(u(s,t))+1)\,\mathrm{sg}(g(u(s,t)) \dot{-} f(u(s,t))),\\
H(f,g,h)(s,t) &= |f(u(s,t))-g(u(s,t))| \dot{-} 1,
\end{aligned}$$

where $u(s,t) = s + (s+1)^2(t+1)$.

The Function $\lambda\xi.\exp(\xi)$ is Conditionally $\mathcal{M}^2$-Computable

It is proved in [SkWeGe 10] that $\min(\exp(\xi), \eta)$ is a uniformly $\mathcal{M}^2$-computable function of ξ and η. Hence there exist $\mathcal{M}^2$-substitutional mappings $F^\circ, G^\circ, H^\circ : \mathbb{T}_1^6 \to \mathbb{T}_1$ such that, whenever (f_1, g_1, h_1) and (f_2, g_2, h_2) are triples from $\mathbb{T}_1^3$ naming the real numbers ξ and η, respectively, then $\min(\exp(\xi), \eta)$ is named by the the triple

$$(F^\circ(f_1, f_2, g_1, g_2, h_1, h_2), G^\circ(f_1, f_2, g_1, g_2, h_1, h_2), H^\circ(f_1, f_2, g_1, g_2, h_1, h_2)).$$

To see the conditional $\mathcal{M}^2$-computability of $\lambda\xi.\exp(\xi)$, we may set

$$\begin{aligned}
E(f,g,h)(s) &= (f(0)+h(0)+1) \dot{-} ((s+1)_1(h(0)+1)+g(0)), \\
F(f,g,h)(s,t) &= F^\circ(f, \lambda x.s+1, g, \lambda x.0, h, \lambda x.0)(t), \\
G(f,g,h)(s,t) &= G^\circ(f, \lambda x.s+1, g, \lambda x.0, h, \lambda x.0)(t), \\
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where $(s+1)_1$ is the exponent of the prime number 3 in $s+1$.

The Function $\lambda\xi.\exp(\xi)$ is Conditionally $\mathcal{M}^2$-Computable

It is proved in [SkWeGe 10] that $\min(\exp(\xi),\eta)$ is a uniformly $\mathcal{M}^2$-computable function of ξ and η. Hence there exist $\mathcal{M}^2$-substitutional mappings $F^\circ, G^\circ, H^\circ : \mathbb{T}_1^6 \to \mathbb{T}_1$ such that, whenever (f_1, g_1, h_1) and (f_2, g_2, h_2) are triples from $\mathbb{T}_1^3$ naming the real numbers ξ and η, respectively, then $\min(\exp(\xi),\eta)$ is named by the the triple

$$(F^\circ(f_1, f_2, g_1, g_2, h_1, h_2), G^\circ(f_1, f_2, g_1, g_2, h_1, h_2), H^\circ(f_1, f_2, g_1, g_2, h_1, h_2)).$$

To see the conditional $\mathcal{M}^2$-computability of $\lambda\xi.\exp(\xi)$, we may set

$$\begin{aligned}
E(f,g,h)(s) &= (f(0)+h(0)+1) \dot{-} ((s+1)_1(h(0)+1)+g(0)),\\
F(f,g,h)(s,t) &= F^\circ(f, \lambda x.s+1, g, \lambda x.0, h, \lambda x.0)(t),\\
G(f,g,h)(s,t) &= G^\circ(f, \lambda x.s+1, g, \lambda x.0, h, \lambda x.0)(t),\\
H(f,g,h)(s,t) &= H^\circ(f, \lambda x.s+1, g, \lambda x.0, h, \lambda x.0)(t),
\end{aligned}$$

where $(s+1)_1$ is the exponent of the prime number 3 in $s+1$.

The Partial Recursive Functions in $\mathbb{N}$ Regarded as Functions in $\mathbb{R}$ are Conditionally $\mathcal{M}^2$–Computable

Let θ be an N-argument partial recursive function. Then θ has a representation of the form

$$\theta(x_1,\dots,x_N) = U(\mu y[\,T(x_1,\dots,x_N,y) = 0]),$$

where $T, U \in \mathcal{M}^2$. To show the conditional $\mathcal{M}^2$–computability of θ, we may set

$$\begin{aligned} E(\bar{f},\bar{g},\bar{h})(s) &= T(x_1,\dots,x_N,s) + \max_{y<s} \overline{\mathrm{sg}}\, T(x_1,\dots,x_N,y), \\ F(\bar{f},\bar{g},\bar{h})(s,t) &= U(s), \\ G(\bar{f},\bar{g},\bar{h})(s,t) &= 0, \\ H(\bar{f},\bar{g},\bar{h})(s,t) &= 0, \end{aligned}$$

where $\bar{f} = f_1,\dots,f_N$, $\bar{g} = g_1,\dots,g_N$, $\bar{h} = h_1,\dots,h_N$, and

$$x_i = \left\lfloor \frac{f_i(1) \dot{-} g_i(1)}{h_i(1)+1} + \frac{1}{2} \right\rfloor, \quad i = 1,\dots,N.$$

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Substitution in Conditionally $\mathcal{F}$-Computable Real Functions

Theorem

Let the class $\mathcal{F}$ contain the addition function and one-argument functions L and R such that $\{(L(s), R(s)) \mid s \in \mathbb{N}\} = \mathbb{N}^2$. Then the substitution operation on real functions preserves conditional $\mathcal{F}$-computability.

As an application, we will show that the function $\theta(\xi) = \ln \xi$ is conditionally $\mathcal{M}^2$-computable. Let us consider the function θ° having domain $\{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_1 > 0, \xi_1\xi_2 \geq 1\}$ and defined by $\theta^\circ(\xi_1, \xi_2) = \ln \xi_1$. This function is uniformly $\mathcal{M}^2$-computable by [SkWeGe 10], hence it is conditionally $\mathcal{M}^2$-computable. On the other hand, $\theta(\xi) = \theta^\circ(\xi, 1/\xi)$ for all $\xi \in \mathrm{dom}(\theta)$.

Since the arctan, arcsin, arccos, sine and cosine functions are shown in [SkWeGe 10] to be uniformly $\mathcal{M}^2$-computable, and so are the sum, difference and product functions, as well as the functions $\sqrt[n]{\xi}$, $n = 2, 3, \ldots$, we may conclude that all elementary functions of calculus are conditionally $\mathcal{M}^2$-computable.

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Local uniform $\mathcal{F}$-computability of the conditionally $\mathcal{F}$-computable functions

Let $N \in \mathbb{N}$ and $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$. The function θ will be called *locally uniformly $\mathcal{F}$-computable* if any point of D has some neighbourhood U such that the restriction of θ to $D \cap U$ is uniformly $\mathcal{F}$-computable.

Theorem

Let for any $a, b \in \mathbb{N}$ the class $\mathcal{F}$ contain the two-argument function whose value at (x, y) is b or y depending on whether or not $x = a$. Let also all one-argument constant functions in $\mathbb{N}$ belong to $\mathcal{F}$. Then all conditionally $\mathcal{F}$-computable real functions are locally uniformly $\mathcal{F}$-computable.

This theorem and a characterization theorem from [Skxx] imply that, under the assumptions about $\mathcal{F}$ in them, if θ is a conditionally $\mathcal{F}$-computable function then each point of $\mathrm{dom}(\theta)$ has some neighbourhood U such that θ is uniformly continuous in $\mathrm{dom}(\theta) \cap U$.

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Some Computable Real Functions which are not Conditionally $\mathcal{F}$-Computable, whatever be the Class $\mathcal{F}$

Let $\theta : \mathbb{R} \setminus \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \to \mathbb{R}$ be defined by

$$\theta(\xi) = \sum_{k=1}^{\infty} \frac{1}{2^k} \sigma\left(\xi - \frac{1}{k}\right),$$

where σ is the restriction of the sign function to $\mathbb{R} \setminus \{0\}$. The function θ is computable in the extended sense of [Gr 55], but there exists no neibourhood U of 0 such that θ is uniformly continuous in $\mathrm{dom}(\theta) \cap U$. By the statement in the last paragraph of the previous slide, θ is not conditionally $\mathcal{F}$-computable for $\mathcal{F} = \bigcup_{m \in \mathbb{N}} \mathbb{T}_m$, therefore it is not conditionally $\mathcal{F}$-computable, whatever be the class $\mathcal{F}$ of total functions in $\mathbb{N}$.

Another similar function is the one obtained from the elementary function $\xi \arctan\left(\tan \frac{1}{\xi}\right)$ by extending it as 0 for $\xi = 0$.

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Uniform $\mathcal{F}$-computability of the locally uniformly $\mathcal{F}$-computable functions with compact domains

Theorem

Let the class $\mathcal{F}$ be closed under substitution, and let $\mathcal{F}$ contain the projection functions, the successor function, the addition function, the function $\lambda xy.x \dot{-} y$ and the function $\lambda xy.x(1 \dot{-} y)$. Then all locally uniformly $\mathcal{F}$-computable real functions with compact domains are uniformly $\mathcal{F}$-computable.

Corollary

Under the assumptions of the above theorem, all conditionally $\mathcal{F}$-computable real functions with compact domains are uniformly $\mathcal{F}$-computable.

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Under the assumptions of the above theorem, all conditionally $\mathcal{F}$-computable real functions with compact domains are uniformly $\mathcal{F}$-computable.

Some Comments

The conditional $\mathcal{F}$-computability of real functions has some similarity in its spirit with the notion of a real function in $\mathcal{F}$ introduced in [TeZi 10] (under some restrictions on the class $\mathcal{F}$) for functions whose domains are open sets. However, there are many essential differences between the two notions. For instance, if $\mathcal{F}$ is the class of the lower elementary functions then the class of the real functions in $\mathcal{F}$ is not closed under substitution, it is not true that it contains all elementary functions of calculus, and there are real functions in $\mathcal{F}$ which are not computable in the extended sense of [Gr 55].

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