Defining Integers and Integral Functions over Global Fields and Their Algebraic Extensions

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June 2011

Outline

The Fields in Play

Definition (Number Fields and Infinite Algebraic Extensions of $\mathbb{Q}$)

Number fields are finite extensions of $\mathbb{Q}$. We will also consider algebraic extensions of $\mathbb{Q}$ of infinite degree.

Definition (Function Fields over Finite Fields of Constants and Their Infinite Algebraic Extensions)

If p is a prime number, $\mathbb{F}_p$ is a finite field of p elements, t is transcendental over $\mathbb{F}_p$, so that $\mathbb{F}_p(t)$ is a rational function field over $\mathbb{F}_p$, then a *function field K of characteristic p* is a finite extension of $\mathbb{F}_p(t)$. The algebraic closure of $\mathbb{F}_p$ in K is called the constant field of K. It is a finite extension of $\mathbb{F}_p$. We will also consider infinite algebraic extensions of $\mathbb{F}_p(t)$.

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The Numbers to Consider

Definition (Algebraic Numbers and Algebraic Integers)

Algebraic numbers are roots of polynomials with coefficients in $\mathbb{Q}$. If α is an algebraic number, then it is an algebraic integer if it satisfies a *monic* irreducible over $\mathbb{Q}$ polynomial with coefficients in $\mathbb{Z}$.

Example

$\sqrt{3}$ is a root of a monic irreducible polynomial $X^2 - 3 = 0$. Thus, $\sqrt{3}$ is an integer. At the same time $\sqrt{\frac{3}{2}}$ is a root of the polynomial $2X^2 - 3$, which is irreducible over $\mathbb{Q}$ and has integer coefficients but is not monic. To make $2X^2 - 3$ monic we have to allow non-integer coefficients, and therefore $\sqrt{\frac{3}{2}}$ is not an algebraic integer.

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Example

$\sqrt{t^2+1}$ is a root of a monic irreducible polynomial $X^2-(t^2+1)=0$. Thus, $\sqrt{t^2+1}$ is an integral function. At the same time $\sqrt{\frac{t+1}{t-1}}$ is a root of the polynomial $(t+1)X^2-(t-1)$, which is irreducible over $\mathbb{F}_p(t)$ and has polynomial coefficients but is not monic. To make $(t+1)X^2-(t-1)$ monic we have to allow rational function coefficients, and therefore $\sqrt{\frac{t+1}{t-1}}$ is not an integral function.

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A Global Perspective

Number fields and function fields over finite fields of constants are jointly called global fields.

Rings of Importance

If K is an algebraic extension of a global field, then the set of all integers (or integral functions) in K form a ring called *the ring of integers (integral functions)* of K. It is also known as the integral closure of $\mathbb{Z}$ or a polynomial ring in K.

The Object of the Game

A Question

If M is a global field or an infinite algebraic extension of a global field, then is there a first-order definition of the ring of integers/ ring of integral functions of the form

$$E_1 x_1 \ldots E_k x_k P(t, x_1, \ldots, x_k) \tag{1}$$

over M, where E_i is either a universal or an existential quantifier, and $P(t, x_1, \ldots, x_k) \in M[t, x_1, \ldots, x_k]$, such that in the formula at most $n \leq k$ variables are in the range of universal quantifiers? (The preferred value of n is 0, of course.)

Outline

Hilbert's Question about Polynomial Equations

Is there an algorithm which can determine whether or not an arbitrary polynomial equation in several variables has solutions in integers?

Using modern terms one can ask if there exists a program taking coefficients of a polynomial equation as input and producing "yes" or "no" answer to the question "Are there integer solutions?".

This problem became known as Hilbert's Tenth Problem

The Answer

This question was answered negatively (with the final piece in place in 1970) in the work of Martin Davis, Hilary Putnam, Julia Robinson and Yuri Matiyasevich.

A General Question

A Question about an Arbitrary Recursive Ring R

Is there an algorithm, which if given an arbitrary polynomial equation in several variables with coefficients in R, can determine whether this equation has solutions in R?

One of the most prominent open questions is the decidability of HTP for $R = \mathbb{Q}$.

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Using Existential Definitions to Solve the Problem

Lemma

If R is a recursive ring containing $\mathbb{Z}$ and such that $\mathbb{Z}$ has an existential definition $p(T, \bar{x})$ over R, i.e. there exists $p(T, x_1, \ldots, x_k) \in R[T, x_1, \ldots, x_k]$ such that for any $t \in R$ we have that

$$t \in \mathbb{Z} \Leftrightarrow \exists \bar{x} \in R^k \, p(T, x_1, \ldots, x_k) = 0,$$

then HTP is not decidable over R.

Proof.

Let $h(T_1, \ldots, T_l)$ be a polynomial with rational integer coefficients and consider the following system of equations.

$$\begin{cases} h(T_1, \ldots, T_l) = 0 \\ p(T_1, \bar{X}_1) = 0 \\ \ldots \\ p(T_l, \bar{X}_l) = 0 \end{cases} \tag{2}$$

It is easy to see that $h(T_1, \ldots, T_l) = 0$ has solutions in $\mathbb{Z}$ iff (2) has solutions in R. Thus if HTP is decidable over R, it is decidable over $\mathbb{Z}$. □

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From Ring to a Field

In general, if K is an algebraic extension of a global field and R is its ring of integers/integral functions having existential (first-order) polynomial definition over K, then the existential (first-order) undecidability of R (in the language of rings) implies the existential (first-order) undecidability of K.

Outline

Results of Julia Robinson

Theorem (1949)

$\mathbb{Z}$ is definable by a first-order formula of the form (1) over $\mathbb{Q}$ with 8 variables in the range of universal quantifiers.

Theorem (1959)

If K is a number field, then O_K is definable over K by a first-order formula (of the form (1) with at least 5 variables in the range of universal quantifiers).

Results of Julia Robinson

Theorem (1949)

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Theorem (1959)

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Results of Julia Robinson

Remark

The definition of the ring of algebraic integers O_K given by Julia Robinson depends on the number field K. It uses explicitly the degree of the field and monic irreducible polynomials of the basis elements.

Results of R. Rumely

Theorem (1979)

If K is a number field then O_K is definable over K, uniformly in K, by a first-order formula (of the form (1) with at least 7 variables in the range of universal quantifiers).

Theorem (1979)

If K is a global function field, then O_K is definable over K, uniformly in K and the characteristic, by a first-order formula (of the form (1) with at least 7 variables in the range of universal quantifiers).

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Results of B. Poonen and J. Koenigsmann

Theorem (Poonen, 2007)

If K is a number field then O_K is definable over K, uniformly in K, by a first-order formula of the form (1) with two variables in the range of universal quantifiers.

Theorem (Koenigsmann, 2009)

$\mathbb{Z}$ is definable over $\mathbb{Q}$ be a first-order formula of the form (1) with only variables in the range of universal quantifiers.

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One Universal Quantifier Definitions over Global Function Fields

Theorem (Work in Progress)

If K is a global function field over a constant field k containing a primitive q-th root of unity for $q \geq 2$, and $t \in K \setminus k$, then $k[t]$ is definable over K by a formula of the form (1) with only universal quantifier. (The only global function field which is not covered by this theorem is a function field over a constant field of size 2.)

Theorem (C. Videla, 1999, 2000)

If K is a number field and M is a pro-p extension of K, then O_M is first-order definable over M.

Infinite Algebraic Extension of Global Fields

Theorem (Work in Progress)

Let K_∞ be an infinite extension of a number field K satisfying the following conditions: for some rational prime $q > 2$, for any rational prime t, there exists a positive constant b_t such that for any number field $E \subset K_\infty$ and any prime $\mathfrak{t}_E$ of E above t we have that $\mathrm{ord}_q(f(\mathfrak{t}_E/t)) \leq b_q$ and $\mathrm{ord}_q(e(\mathfrak{t}_E/t)) \leq b_q$. In this case O_{K_∞} is first-order definable over K_∞.

Infinite Algebraic Extension of Global Fields

Corollary

Let K_∞ be a normal infinite extension of a number field K satisfying the following condition: for some rational prime q and any number field $E \subset K_\infty$ such that $K \subset E$ and E/K is Galois, we have that $([E : K], p) = 1$. In this case O_{K_∞} is first-order definable over K_∞.

Corollary

Let q be a rational prime, let $\{\ell_i\}$ be a sequence of rational prime numbers not containing q and such that no $\ell_i \equiv 1 \mod q$. Let $\xi_{\ell_i^j}$ be an ℓ_i^j-th primitive root of unity. If $K_\infty = \mathbb{Q}(\xi_{\ell_i^j}, i, j \in \mathbb{Z}_{>0})$, then O_{K_∞} is first-order definable over K_∞.

Outline

The Role of Denominators

What is the difference between a ring of integers/ integral functions and the field?

One difference is presence of denominators. So we are looking for a first order statement saying "no denominators are allowed".

Decomposing denominators

We think of a denominator as a product of primes in the sense we will make precise below. The statement we are looking for will say something like "no prime can divide the denominator".

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Outline

Primes of Number Fields

Definition

A prime of a number field K is a prime ideal of O_K.

In the case $K = \mathbb{Q}$ and $O_K = \mathbb{Z}$ the primes are the prime numbers identified with the prime ideals they generate in $\mathbb{Z}$.

Order at a Prime

Order at a Prime in a Number Field

If K is a number field, $x \neq 0$ and $x \in O_K$, then for any prime $\mathfrak{p}$ of K there exists a non-negative integer m such that $x \in \mathfrak{p}^m$ but $x \notin \mathfrak{p}^{m+1}$. We call m the order of x at $\mathfrak{p}$ and write $m = \operatorname{ord}_{\mathfrak{p}} x$. If $y \in K$ and $y \neq 0$, we write $y = \frac{x_1}{x_2}$, where $x_1, x_2 \in O_K$ with $x_1 x_2 \neq 0$, and define $\operatorname{ord}_{\mathfrak{p}} y = \operatorname{ord}_{\mathfrak{p}} x_1 - \operatorname{ord}_{\mathfrak{p}} x_2$. This definition is not dependent on the choice of x_1 and x_2 which are of course not unique. We define $\operatorname{ord}_{\mathfrak{p}} 0 = \infty$ for any prime $\mathfrak{p}$ of K. If $\operatorname{ord}_{\mathfrak{p}} y \geq 0$ then we say that y *is integral at* $\mathfrak{p}$.

Example using a rational number

Given an element x of a global field K we can think of the primes at which x has a negative order as being in the "denominator". In the case of $K = \mathbb{Q}$ we have the usual definition of the denominator. Let $x = \frac{5^2 3^4}{7^3}$. In this case $\mathrm{ord}_5 x = 2, \mathrm{ord}_3 x = 4, \mathrm{ord}_7 x = -3$ and 7 is in the "denominator". Also for any $p \neq 3, 5, 7$ we have that $\mathrm{ord}_p x = 0$.

Existential Definition of Integrality at a Finite Number of Primes for Number Field

Proposition (Julia Robinson)

If K is a number field, $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ is a finite collection of primes of K, then the set $\{x \in K : \operatorname{ord}_{\mathfrak{p}_i} x \geq 0\}$ is existentially definable over K.

Integral Functions and Primes of Global Function Fields

A prime of a function field K is a prime ideal of O_K or a prime ideal of $O_{K,\infty}$– the integral closure of $\mathbb{F}_p[\frac{1}{t}]$, containing $\frac{1}{t}$. The prime ideals of $O_{K,\infty}$ are referred to as *infinite primes*.

Order at a Prime over Global Function Fields

Order at a Prime from O_K over a Function Field

If K is a global function field, $x \neq 0$ and $x \in O_K$, then for any prime $\mathfrak{p}$ of K originating in O_K there exists a non-negative integer m such that $x \in \mathfrak{p}^m$ but $x \notin \mathfrak{p}^{m+1}$. We call m the order of x at $\mathfrak{p}$ and write $m = \operatorname{ord}_{\mathfrak{p}} x$. If $y \in K$ and $y \neq 0$, we write $y = \frac{x_1}{x_2}$, where $x_1, x_2 \in O_K$ with $x_1 x_2 \neq 0$, and define $\operatorname{ord}_{\mathfrak{p}} y = \operatorname{ord}_{\mathfrak{p}} x_1 - \operatorname{ord}_{\mathfrak{p}} x_2$. This definition is not dependent on the choice of x_1 and x_2 which are of course not unique. We define $\operatorname{ord}_{\mathfrak{p}} 0 = \infty$ for any prime $\mathfrak{p}$ of O_K.

Order at a Prime from $O_{K,\infty}$ over a Function Field

The order at the primes which are ideals of $O_{K,\infty}$ are defined in the analogous manner with $O_{K,\infty}$ substituting for O_K.

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Primes of a Rational Function Field

In the case $K = \mathbb{F}_p(t)$ all but one prime correspond to irreducible polynomials in t and the remaining (infinite) prime corresponds to the degree of polynomials. For example, consider $x = \frac{t^2+1}{t-1}$ over $\mathbb{F}_p$ where -1 is not a square. Let $\mathfrak{p}_1$ correspond $t^2 + 1$, $\mathfrak{p}_2$ correspond to $t - 1$, $\mathfrak{p}_\infty$ correspond to degree. In this case,

$$\operatorname{ord}_{\mathfrak{p}_1} x = 1,$$

$$\operatorname{ord}_{\mathfrak{p}_2} x = -1,$$

$$\operatorname{ord}_{\mathfrak{p}_\infty} x = \operatorname{ord}_{\mathfrak{p}_\infty}(t^2 + 1) - \operatorname{ord}_{\mathfrak{p}_\infty}(t - 1) = -2 - (-1) = -1.$$

Integrality at Finitely Many Primes over Global Function Fields

Proposition (R. Rumely)

If K is a global function field, $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ is a finite collection of primes of K, then the set $\{x \in K : \mathrm{ord}_{\mathfrak{p}_i} x \geq 0, i = 1, \ldots, m\}$ is existentially definable over K.

Outline

A set of elements with poles of order divisible by a prescribed prime

Definition

Given a global field K and a rational prime q, let Divq(K) be a subset of all the elements u of K such that for all primes $\mathfrak{p}$ of K, either $\mathrm{ord}_{\mathfrak{p}}\, u \geq 0$ or $\mathrm{ord}_{\mathfrak{p}}\, u \equiv 0 \mod q$.

The order of the sum

Non-archimedean triangular inequality

If K is a global field, $x, y \in K$, $\mathfrak{p}$ is a prime of K, then

$$\operatorname{ord}_{\mathfrak{p}}(x+y) \geq \min(\operatorname{ord}_{\mathfrak{p}} x, \operatorname{ord}_{\mathfrak{p}} y).$$

Further, if $\operatorname{ord}_{\mathfrak{p}} x \neq \operatorname{ord}_{\mathfrak{p}} y$, then

$$\operatorname{ord}_{\mathfrak{p}}(x+y) = \min(\operatorname{ord}_{\mathfrak{p}} x, \operatorname{ord}_{\mathfrak{p}} y).$$

Example

$$\operatorname{ord}_3(3+3) = \operatorname{ord}_3 6 = \operatorname{ord}_3 3 = 1$$

$$\operatorname{ord}_3(3+9) = \operatorname{ord}_3 12 = \operatorname{ord}_3 3 = 1$$

$$\operatorname{ord}_3(3+6) = \operatorname{ord}_3 9 = 2$$

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Other Properties of Order at a Prime in a Number Field

Let K be a number field.

- $\forall x, y \in K$ and any prime $\mathfrak{p}$ of K we have that $\operatorname{ord}_{\mathfrak{p}} xy = \operatorname{ord}_{\mathfrak{p}} x + \operatorname{ord}_{\mathfrak{p}} y$.
- $\forall x, y \in K$ and any prime $\mathfrak{p}$ of K we have that $\operatorname{ord}_{\mathfrak{p}} \frac{x}{y} = \operatorname{ord}_{\mathfrak{p}} x - \operatorname{ord}_{\mathfrak{p}} y$
- If K is a number field and $z \in O_K$, then for all primes of $\mathfrak{p}$ of K, we have that $\operatorname{ord}_{\mathfrak{p}} z \geq 0$.
- If ξ is a unit of O_K (that is $\xi^{-1} \in O_K$), then $\operatorname{ord}_{\mathfrak{p}} \xi = 0$ for all primes $\mathfrak{p}$ of K.
- For any $n \in \mathbb{Z}$ we have $\operatorname{ord}_{\mathfrak{p}} x^n = n \operatorname{ord}_{\mathfrak{p}} x$.
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Let K be a number field.

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- $\forall x, y \in K$ and any prime $\mathfrak{p}$ of K we have that $\operatorname{ord}_{\mathfrak{p}} \frac{x}{y} = \operatorname{ord}_{\mathfrak{p}} x - \operatorname{ord}_{\mathfrak{p}} y$
- If K is a number field and $z \in O_K$, then for all primes of $\mathfrak{p}$ of K, we have that $\operatorname{ord}_{\mathfrak{p}} z \geq 0$.
- If ξ is a unit of O_K (that is $\xi^{-1} \in O_K$), then $\operatorname{ord}_{\mathfrak{p}} \xi = 0$ for all primes $\mathfrak{p}$ of K.
- For any $n \in \mathbb{Z}$ we have $\operatorname{ord}_{\mathfrak{p}} x^n = n \operatorname{ord}_{\mathfrak{p}} x$.
- For any prime $\mathfrak{p}$ of K, there exists $b \in K$ such that $\operatorname{ord}_{\mathfrak{p}} b = 1$.

Using Divisibility of Order

Proposition

If K is a number field then O_K - the ring of integers of K can be defined using the following formula.

$$z \in O_K \iff \forall b \in K : z^q b + b^q \in Divq(K)$$

Proof of the Divisibility of Order Proposition

Suppose $z \in O_K$ and let $b \in K$, $\mathfrak{p}$ a prime of K. We consider two cases: $\operatorname{ord}_{\mathfrak{p}} b \geq 0$ and $\operatorname{ord}_{\mathfrak{p}} b < 0$. Note that $z \in O_K \Rightarrow \operatorname{ord}_{\mathfrak{p}} z \geq 0$.

- $\operatorname{ord}_{\mathfrak{p}} b \geq 0$:

$$\operatorname{ord}_{\mathfrak{p}} bz^q + b^q \geq \min(\operatorname{ord}_{\mathfrak{p}} b + \operatorname{ord}_{\mathfrak{p}} z^q, b^q) \geq 0$$

- $\operatorname{ord}_{\mathfrak{p}} b < 0$: Observe that in this case

$$\operatorname{ord}_{\mathfrak{p}} b + \operatorname{ord}_{\mathfrak{p}} z^q \geq \operatorname{ord}_{\mathfrak{p}} b > \operatorname{ord}_{\mathfrak{p}} b^q.$$

Hence

$$\operatorname{ord}_{\mathfrak{p}} bz^q + b^q = \min(\operatorname{ord}_{\mathfrak{p}} b + \operatorname{ord}_{\mathfrak{p}} z^q, b^q) = \operatorname{ord}_{\mathfrak{p}} b^q = q \operatorname{ord}_{\mathfrak{p}} b.$$

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Proof of the Divisibility of Order Proposition, continued

Suppose now that $z \notin O_K \Rightarrow \mathrm{ord}_{\mathfrak{p}}\, z \leq -1$ for some prime $\mathfrak{p}$ of K. Let $b \in K$ be such that $\mathrm{ord}_{\mathfrak{p}}\, b = -1$ and note that in this case

$$\mathrm{ord}_{\mathfrak{p}}\, z^q b = q\,\mathrm{ord}_{\mathfrak{p}}\, z + \mathrm{ord}_{\mathfrak{p}}\, b \leq -(q+1) < -q = \mathrm{ord}_{\mathfrak{p}}\, b^q,$$

so that

$$\mathrm{ord}_{\mathfrak{p}}(z^q b + b^q) = \min(\mathrm{ord}_{\mathfrak{p}}\, z^q b, \mathrm{ord}_{\mathfrak{p}}\, b^q) =$$

$$\mathrm{ord}_{\mathfrak{p}}\, z^q b = q\,\mathrm{ord}_{\mathfrak{p}}\, z - 1 \equiv -1 \mod q.$$

Hence we conclude that $(z^q b + b^q)$ has a negative order at $\mathfrak{p}$ and the order is not divisible by q. Thus, $(z^q b + b^q) \notin Divq(K)$

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What does it all mean?

Corollary

If for some rational prime number q we have that $Divq$ is existentially definable over a number field K, then O_K has a one-universal-quantifier definition over K.

In his paper Jochen Königsmann essentially constructed an existential definition of $Div2(\mathbb{Q})$ and thus produced a one-universal-quantifier definition of $\mathbb{Z}$ over $\mathbb{Q}$.

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Extensions to Number Fields

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In Pursuit of Uniformity and Other Primes

Theorem

If K is a global field of characteristic $p \geq 0$, $q \neq p$ a rational prime number, ξ_q a primitive q-th root of unity, and $[K(\xi_q) : K] = d$, then $Divq(K)$ is definable in the following way

$$\{u \in K | \forall w_1 \ldots \forall w_d \exists z_1 \ldots \exists z_k P(u, w_1, \ldots, w_d, z_1, \ldots, z_k) = 0\},$$

where $P(u, z, z_1, \ldots, z_k) \in K[u, z, z_1, \ldots, z_k]$. Further, this definition is uniform across all the fields satisfying the conditions above.

Corollary

$Div2(K)$ is definable using one quantifier uniformly across all global fields K of characteristic not equal to 2.

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What Are the Benefits of Being Uniform and Using Other Primes?

- Uniformity and using arbitrary primes leads to first-order definitions of integers and integral functions over infinite extensions
- Using $q \neq 2$ allows us to consider fields of all characteristic.

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Some Remarks on Definitions over Global Function Fields

- If K is a global function field, then the set $P(K) = \{(x, x^{p^s}) | x \in K, s \in \mathbb{Z}_{>0}\}$ is existentially definable over K (in the language of rings). This definition is *not uniform* across different characteristics and across fields of the same characteristic. (It essentially dependens on an invariant of a function field called the *genus*.)
- The construction of a one-universal-quantifier definition of integral functions over a global field uses a one-universal-quantifier definition of the divisibility of order and the p-th powers equations. Thus it is not uniform across different fields as explained above.
- Using a function field version of the equivalence $z \in O_K \iff \forall b \in K : z^2 b + b^2 \in \mathrm{Div2}(K)$, where $\mathrm{Div2}(K)$ is defined using one universal quantifier, one can give a uniform definition of integral function across all global fields of characteristic greater than 2.
- It is possible that using Königsmann's ideas for existential definition of $Div2(\mathbb{Q})$ one can construct a more uniform one-universal-quantifier definition of integral functions over global function fields.

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