Complexity Issues for Preorders on Finite Labeled Forests

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CiE-2011, Sofia, June 28, 2011

Weihrauch Reducibilities

In [Wei92, Wei00, Hir93] K. Weihrauch and M. Hirsh introduced some notions of reducibility for functions on topological spaces which turned out useful for understanding non-computability and non-continuity of interesting decision problems in computable analysis [HW94, Her96, BG11a] and constructive mathematics [BG11]. In particular, the following three notions of reducibilities between functions $f, g : X \to Y$ on topological spaces were introduced:

- $f \leq_0 g$ iff $f = g \circ H$ for some continuous function $H : X \to X$;
- $f \leq_1 g$ iff $f = F \circ g \circ H$ for some continuous functions $H : X \to X$ and $F : Y \to Y$;
- $f \leq_2 g$ iff $f(x) = F(x, gH(x))$ for some continuous functions $H : X \to X$ and $F : X \times Y \to Y$.

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Weihrauch Reducibilities

The notions are nontrivial even for the case of discrete spaces $Y = k = \{0, \ldots, k-1\}$ with $k < \omega$ points (we call functions $f : X \to k$ *k-partitions of X*). E.g., for $k = 2$ the relation $\leq_0$ coincides with the classical Wadge reducibility.

Weihrauch Degrees and Labeled Forests

In [Her93, Her96] P. Hertling gave a useful "combinatorial" characterisation of some important initial segments of the degree structures under Weihrauch reducibilities on k-partitions of the Baire space (note that the Baire space is important because it is commonly used in computable analysis for representing many other spaces of interest).

Namely, he introduced preorders $\leq_0, \leq_1, \leq_2$ on the set $\mathcal{F}_k$ of finite k-labeled forests (precise definitions are given in the next section) such that, for each $i \leq 2$, the structure of the topological $\leq_i$-degrees of the initial segment is isomorphic to the quotient-poset of $(\mathcal{F}_k; \leq_i)$.

k-Posets and Boolean Hierarchy of k-Partitions

The structure $(\mathcal{F}_k; \leq_0)$ and its extension to the structure $(\mathcal{P}_k; \leq_0)$ of finite k-labeled posets are also important due to their close relationship to the Boolean hierarchy of k-partitions that extends the classical Boolean (or difference) hierarchy of sets [KW00, Kos05, Sel04] and to some other fields of discrete mathematics like clones of functions on k [Leh08, KL10].

The mentioned results motivated the study of $(\mathcal{F}_k; \leq_0)$ and $(\mathcal{P}_k; \leq_0)$ in a series of publications.

Some Known Facts

In [KS06] it was shown that for any $k \geq 3$ the first-order theory of the quotient-poset of $(\mathcal{F}_k; \leq_0)$ is undecidable. It is even computably isomorphic to first-order arithmetic [KS07].
In [KSZ10] the same was shown for $\leq_1$ and $\leq_2$.
In [KS09] a complete definability theory for the quotient-poset of $(\mathcal{F}_k; \leq_0)$ was developed.
According to [Leh08], the quotient-poset of $(\mathcal{P}_k; \leq_0)$ is universal, i.e., it contains any countable poset as a substructure. In contrast, the quotient-poset of $(\mathcal{F}_k; \leq_i)$ is wqo for $i = 0, 1, 2$.

Objectives of This Work

The naming systems $(\mathcal{F}_k; \leq_i)$ and $(\mathcal{P}_k; \leq_i)$ are obviously computable, and the natural next step in understanding their computational properties is to look at their complexity.

In [KL10] it is shown that the relation $\leq_0$ on $\mathcal{P}_k$ is NP-complete.

In this paper we answer some natural complexity questions about the relations $\leq_0, \leq_1, \leq_2$ on $\mathcal{F}_k$, and also on the set $\mathcal{F}_\omega$ of finite ω-labeled posets.

Note that in [Her96] it is shown that the structure of $\leq_2$-degrees of functions $f : \mathbb{B} \to \omega$ such that all $f^{-1}(k)$ are finite Boolean combinations of open sets is isomorphic to the quotient-poset of $(\mathcal{F}_\omega; \leq_2)$.

Main Results

Theorem

The relation $\leq_0$ is polynomial-time computable on $\mathcal{F}_k$ for $k < \omega$ and on $\mathcal{F}_\omega$.

Theorem

The relations $\leq_1$ and $\leq_2$ are polynomial-time computable on $\mathcal{F}_k$ for $k < \omega$, and they are NP-complete on $\mathcal{F}_\omega$.

k-Forests

Let $2 \leq k < \omega$. A k-labeled forest (or just a k-forest) is a triple $(P; \leq, c)$ formed by a finite forest $(P; \leq)$ and a labeling $c : P \to k$.
ω-forests are defined in the same way
Let $\mathcal{F}_k$ be the set of finite k-forests $(P; \leq, c)$ where $P \subseteq \omega$. The set $\mathcal{F}_\omega$ is defined in the same way.

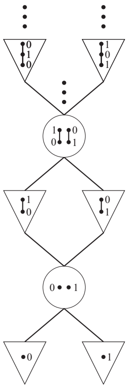
Morphisms

- A 0-*morphism* $f : (P; \leq_P, c_P) \to (Q; \leq_Q, c_Q)$ between k-forests is a monotone function $f : (P; \leq_P) \to (Q; \leq_Q)$ respecting the labelings, i.e. satisfying $c_P = c_Q \circ f$.
- A 1-*morphism* $f : (P; \leq_P, c_P) \to (Q; \leq_Q, c_Q)$ is a monotone function $f : (P; \leq_P) \to (Q; \leq_Q)$ for which there exists a mapping $g : k \to k$ such that $c_P = g \circ c_Q \circ f$.
- A 2-*morphism* $f : (P; \leq_P, c_P) \to (Q; \leq_Q, c_Q)$ is a monotone function $f : (P; \leq_P) \to (Q; \leq_Q)$ such that

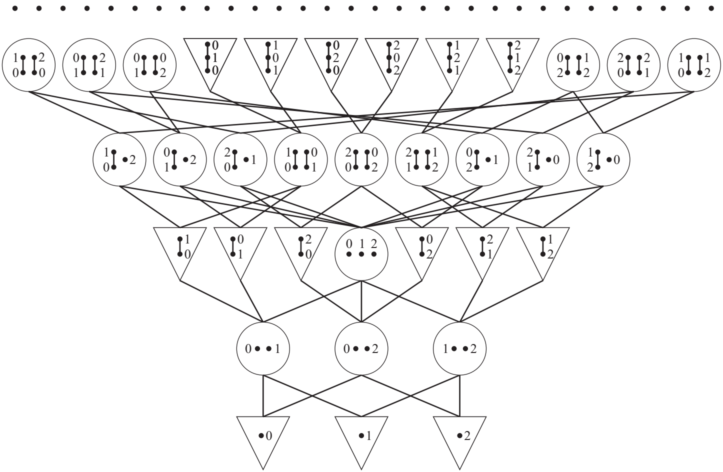
$$\forall x, y \in P((x \leq_P y \wedge c_P(x) \neq c_P(y)) \to c_Q(f(x)) \neq c_Q(f(y))).$$

Preorders on k-Forests

We say that a k-forest P is 0-*morphic* (resp. 1-*morphic*, 2-*morphic*) to a k-forest Q, denoted $P \leq_0 Q$ (respectively $P \leq_1 Q$, $P \leq_2 Q$), iff there exists a 0-morphism (resp. 1-morphism, 2-morphism) $f : P \to Q$.
The relations $\leq_0, \leq_1, \leq_2$ are preorders on $\mathcal{F}_\omega$.
It is easy to prove that any 0-morphism is a 1-morphism and any 1-morphism is a 2-morphism in turn. Therefore $\leq_0$ implies $\leq_1$ and $\leq_1$ implies $\leq_2$.



Picture 1: An initial segment of $\mathcal{F}_2$.



Picture 2: An initial segment of $\mathcal{F}_3$.

The Complexity of $\leq_0$

Theorem

There is a Turing machine which, given two k-labeled forests F and G, checks in time $O(m+n+m^2n+mn^2)$ (where $m := |F|$ and $n := |G|$) whether $F \leq_0 G$ or not, for any $k < \omega$. The same is true for ω-labeled forests (where now m resp. n are the lengths of representations of F and G, respectively).

Thus, the relation $\leq_0$ is computable in cubic time on $\mathcal{F}_k$, for any $k < \omega$, and on $\mathcal{F}_\omega$.
Also some other natural relations and functions on $\mathcal{F}_k$ are computable in polynomial time (in particular the function, relating to any k-forest the minimal h-equivalent k-forest).

Proof Sketch

Estimation follows from analysis of the following recursive algorithm:

1. If $|F| = 0$ then $F \leq_0 G$ is true.
2. If $|F| \geq 1$ and $|G| = 0$ then $F \leq_0 G$ is false.
3. If $F = p_i(F_0)$ and $G = p_i(G_0)$ then $F \leq_0 G$ iff $F_0 \leq_0 G$.
4. If $F = p_i(F_0)$, $G = p_j(G_0)$, and $i \neq j$ then $F \leq_0 G$ iff $F \leq_0 G_0$.
5. If $G = G_0 \sqcup G_1$ then $F \leq_0 G$ iff $F \leq_0 G_0 \vee F \leq_0 G_1$.
6. If $F = F_0 \sqcup F_1$ then $F \leq_0 G$ iff $F_0 \leq_0 G \wedge F_1 \leq_0 G$.

The Complexity of $\leq_1$

Lemma

$\leq_1$ and $\leq_2$ coincide on $\mathcal{C}_\omega$ and also, for each $k < \omega$, on $\mathcal{C}_k$.

Proposition

The relations $\leq_1$ and $\leq_2$ on $\mathcal{C}_\omega$ are NP-hard.

Hint. We relate in polynomial time to any 3-CNF $C = C(x_0, \ldots, x_n)$ words u, v over the alphabet $\{0, 1, x_0, \bar{x}_0 \ldots, x_n, \bar{x}_n\})$ such that C is satisfiable iff $u \leq_2 v$.

Theorem

For any $k < \omega$, the relation $\leq_1$ on $\mathcal{F}_k$ is computable in cubic time. The relation $\leq_1$ on $\mathcal{F}_\omega$ is NP-complete.

Proof of Theorem 5

Let $k < \omega$. For any $G = (Q; \leq, c) \in \mathcal{F}_k$ and $f : k \to k$, we define $G_f := (Q; \leq, f \circ c)$. From the definition of $\leq_1$ we observe that $F \leq_1 G$ iff there exists a function $f : k \to k$ such that $F \leq_0 G_f$. Let $\{f_1, \ldots, f_{k^k}\}$ be an enumeration without repetition of $\{f \mid f : k \to k\}$. Since $G \mapsto (G_{f_1}, \ldots, G_{f_{k^k}})$ is computable in linear time and $\leq_0$ is computable in cubic time, $\leq_1$ on $\mathcal{F}_k$ is computable in cubic time.

For $\mathcal{F}_\omega$, guess a function $f : c_Q(Q) \to c_P(P)$ and a function $\varphi : P \to Q$ and check (in polynomial time) whether φ is a 0-morphism from $(P; \leq_P, c_P)$ to $(Q; \leq_Q, f \circ c_Q)$.

The Complexity of $\leq_2$

Theorem

For any $k < \omega$, the relation $\leq_2$ on $\mathcal{F}_k$ is computable in time $O(n^{k+3})$. The relation $\leq_2$ on $\mathcal{F}_\omega$ is NP-complete.

Proof Sketch. For $F = (P; \leq_P, c_P), G = (Q; \leq_Q, c_Q) \in \mathcal{F}_k$, and $f :\subseteq k \to k$, let $F \leq_2^f G$ mean that there is a 2-morphism $\varphi : F \to G$ such that for all $x \in P$, if $c_Q(\varphi(x)) \in dom(f)$ then $c_P(x) = f(c_Q(\varphi(x)))$. Note that $F \leq_2 G$ is equivalent to $F \leq_2^{f_\emptyset} G$ where $f_\emptyset = \emptyset$.
It suffices to show that the relation $\leq_2^f$ is computable in time $O(n^{3+k-|dom(f)|})$, for any function $f :\subseteq k \to k$. This follows from:

The Complexity of $\leq_2$

1. If $|F| = 0$ then $F \leq_2^f G$ is true.
2. If $|F| \geq 1$ and $|G| = 0$ then $F \leq_2^f G$ is false.
3. If $F = p_i(F_0)$, $G = p_j(G_0)$, $j \in dom(f)$, and $i = f(j)$ then $F \leq_2^f G$ iff $F_0 \leq_2^f G$.
4. If $F = p_i(F_0)$, $G = p_j(G_0)$, $j \in dom(f)$, and $i \neq f(j)$ then $F \leq_2^f G$ iff $F \leq_2^f G_0$.
5. If $F = p_i(F_0)$, $G = p_j(G_0)$, and $j \notin dom(f)$ then $F \leq_2^f G$ iff $(F \leq_2^f G_0 \vee (F_0 \leq_2^g G$ where g is defined by $dom(g) := dom(f) \cup \{j\}$, and by $g(l) := f(l)$ for $l \in dom(f)$, and $g(j) := i))$.
6. If $G = G_0 \sqcup G_1$ then $F \leq_2^f G$ iff $F \leq_2^f G_0 \vee F \leq_2^f G_1$.
7. If $F = F_0 \sqcup F_1$ then $F \leq_2^f G$ iff $F_0 \leq_2^f G \wedge F_1 \leq_2^f G$.

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