CiE 2011, Sofia, Bulgaria

Special Session Computability in Analysis, Algebra, and Geometry

Orders on Structures and Structure of Orders

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- Magma is a nonempty set with a binary operation: $(M, \cdot)$

- A linear (partial) ordering $<$ of the domain M is a (partial) *left-order* on the structure $(M, \cdot)$ if it is left invariant with respect to $\cdot$:
  $(\forall x, y, z)[x < y \Rightarrow z \cdot x < z \cdot y]$

- $<$ is a *bi-order* (*order*) on the structure if
  $(\forall x, y, z)[x < y \Rightarrow z \cdot x < z \cdot y \wedge x \cdot z < y \cdot z]$

- $LO(M)$ the set of left orders on M
  $RO(M)$ the set of right orders on M
  $BiO(M)$ the set of bi-orders on M

- Given a left order $<_l$ on a group G, we have a right order $<_r$:
  $x <_r y \Leftrightarrow y^{-1} <_l x^{-1}$

- G is left-orderable group $\Rightarrow$ G is *torsion-free*
  torsion-free: $(\forall x \in G - \{e\})[order(x) = \infty]$
  $e < x \Rightarrow x < x^2 < \cdots < x^n$

- (Levy)
  G is abelian and torsion-free $\Rightarrow$ G is orderable

- (Kokorin and Kopytov)
  Every torsion-free nilpotent group is orderable.

- Not every torsion-free group is left-orderable.

- Let $<$ be a partial left order on a group G

  Positive partial cone: $P = \{a \in G \mid a \geq e\}$

  Negative partial cone: $P^{-1} = \{a \in G \mid a \leq e\}$

  1. $PP \subseteq P$ (P *sub-semigroup* of G)
  2. $P \cap P^{-1} = \{e\}$ (P *pure*)

- P with 1 & 2 defines a partial left order $\leq_P$ on G:

$$x \leq_P y \Leftrightarrow x^{-1}y \in P$$

$x \leq_P y \Rightarrow x^{-1}y \in P$

$\Rightarrow x^{-1}z^{-1}zy = (zx)^{-1}(zy) \in P$

$\Rightarrow zx \leq_P zy$

- P with 1 & 2 defines a *left order* if

3. $P \cup P^{-1} = G$ (P *total*)

- P with 1, 2 & 3 defines a *bi-order* if:

  4. $(\forall g \in G)[g^{-1}Pg \subseteq P]$ (P *normal*)

  bi-order $>$: let $g \in G$
  $x > e \Rightarrow g^{-1}xg > g^{-1}eg = e$

  P normal: let $x \leq_P y$, $z \in G$

  right invariant: $x^{-1}y \in P \Rightarrow z^{-1}x^{-1}yz \in P$
  $(xz)^{-1}yz \in P \Rightarrow xz \leq_P yz$

- For groups, orders often identified with their positive cones.

- Example: $G = \mathbb{Z} \oplus \mathbb{Z}$ bi-orderable with a positive cone

$$P = \{(a, b) \mid 0 < a \vee (a = 0 \wedge 0 \leq b)\}.$$

- Fundamental group of Klein bottle
  $G = \left\langle x, y \mid xyx^{-1}y = e \right\rangle$ left-orderable, but not bi-orderable.

  Positive cone $P = \{x^n y^m \mid n > 0 \vee (n = 0 \wedge m \geq 0)\}$
  defines a left order on G.

  If $<$ bi-order on G, then $y > e$ or $y < e$

  $y > e \Rightarrow y^{-1} = xyx^{-1} > e$

  $y < e \Rightarrow y^{-1} = xyx^{-1} < e$,
  contradiction.

- A magma $(Q, *)$ is a *quandle* if:

  1. $(\forall a)[a * a = a]$ (idempotence);

  2. for every $b \in Q$, the mapping $*_b : Q \longrightarrow Q$ defined by

  $$*_b(a) = a * b$$

  is bijective;

  3. $(\forall a, b, c)[(a * b) * c = (a * c) * (b * c)]$ (right self-distributivity).

- A quandle Q is called *trivial* if the operation $*$ is defined by

  $$(\forall a, b)[a * b = a].$$

  Every linear ordering of elements of Q is right invariant.

- For a group G, the *conjugate* quandle Conj(G) is one with domain G and the operation $*$ given by $a * b = b^{-1}ab$.
  Then every bi-order on G induces a right order on Conj(G).

  Let P be a bi-order on G. Then

$$(\forall x, c)[(e, x) \in P \Rightarrow (e, c^{-1}xc) \in P)]$$

  Using P, we define R on Conj(G) as

$$(\forall a, b)[(a, b) \in R \Leftrightarrow (e, a^{-1}b) \in P],$$

  where e is the identity of G.

  The order R is right invariant because for $(a, b) \in R$ and $c \in$ Conj(G),

$$(e, (a * c)^{-1}(b * c)) = (e, (c^{-1}a^{-1}c)(c^{-1}bc)) = (e, c^{-1}(a^{-1}b)c) \in P.$$

  Since $(e, a^{-1}b) \in P$, we have $(a * c, b * c) \in R$.

- Not all right orders on Conj(G) are induced by bi-orders on G.
  It is possible to have $BiO(G) = \emptyset$,
  while $RO(\text{Conj}(G)) \neq \emptyset$.

  Let G be an abelian group with torsion.
  Then $BiO(G) = \emptyset$, but Conj(G) is a trivial quandle,
  so it admits many right orders.

- n-*quandle* Q_n: $(\forall a, b)[b * a^{*n} = b]$, where

$$b * a^{*n} = (\dots(b * a) * a) * \dots * a) * a \text{ with } n \ a\text{'s}$$

  For $n = 2$ we have *involutive* quandle Q_2: for every group define

$$b * a = ab^{-1}a$$

  Then $RO(Q_n) = \emptyset$ unless $n = 1$.

- *Topology* defined on $LO(M)$ by subbasis $\{S_{(a,b)}\}_{(a,b)\in(M\times M)-\Delta}$ where $\Delta = \{(a,a) \mid a \in M\}$:

$$S_{(a,b)} = \{R \in LO(M) \mid (a,b) \in R\}.$$

- (Dabkowska, Dabkowski, Harizanov, Przytycki, Veve, 2007)

  Let M be a magma with cardinality $|\mathcal{M}| = \mathfrak{m} \geq \aleph_0$.
  Then $LO(M)$ is a compact space.

  By Vedenissoff's theorem, $LO(M)$ can be homeomorphically embedded into the Cantor cube $\{0,1\}^{\mathfrak{m}}$.
  Moreover, $LO(M)$ is a closed subspace of the Cantor cube $\{0,1\}^{\mathfrak{m}}$.

- If M is a countable magma, then $LO(M)$ is metrizable.

- If $M = G$ is a group, we showed how we could also use Conrad's theorem to establish that $LO(G)$ is compact.

- (Conrad, 1959) A partial left order P can be extended to a total left order on G iff for every $\{x_1, ..., x_n\} \subset G \backslash \{e\}$ there are $\epsilon_1, ..., \epsilon_n$, $\epsilon_i \in \{1, -1\}$, such that

$$e \notin sgr((P \backslash \{e\}) \cup \{x_1^{\epsilon_1}, ..., x_n^{\epsilon_n}\}),$$

where $sgr(A)$ is the sub-semigroup of G generated by A.

- For a countable group G, $LO(G) \neq \emptyset$ is homeomorphic to the Cantor set iff for any sequence $(a_0, b_0), ..., (a_{k-1}, b_{k-1})$, $S_{(a_0,b_0)} \cap \cdots \cap S_{(a_{k-1},b_{k-1})}$ is either empty or infinite.

- (Sikora, 2004)
  The space $LO(\mathbb{Z}^n)$ for $n > 1$ is homeomorphic to the Cantor set.

  (Dabkowska, 2006)
  The space $LO(\mathbb{Z}^\omega)$ is homeomorphic to the Cantor set.

- (Linnell, 2006)
  The space of left orders of a countable left-orderable group is either finite or contains a homeomorphic copy of the Cantor set.

There are countable groups with infinitely countably many bi-orders.

- (Solomon, 1998)
  For every bi-orderable computable group G, there is
  a computable binary tree $\mathcal{T}$ and a Turing degree preserving bijection
  from $BiO(G)$ to the set of all infinite paths of $\mathcal{T}$.

- Hence, by the Low Basis Theorem of Jockusch and Soare,
  $\mathcal{T}$ has a *low* infinite path.

  Recall that a set X and its Turing degree $\mathbf{x}$ are *low* if $\mathbf{x}' = \mathbf{0}'$.

  Hence $BiO(G)$ contains an order of *low* Turing degree.

- (Metakides and Nerode, 1979)
  The sets of orders on computable fields are in
  exact correspondence to the sets of Π^0_1 classes.

- (Downey and Kurtz, 1986)
  There is a computable torsion-free abelian group with no computable order.

- (Dobrica, 1983)
  Every computable torsion-free abelian group is isomorphic to a computable group
  with a computable basis.

- Every computable torsion-free abelian group is isomorphic to a computable group
  with a computable order.

- (Harizanov, Knight, Lange, Puzarenko, Solomon, Wallbaum, 2011)

  Let be $\mathcal{F}_\infty$ be the free group of rank $\aleph_0$.

  (i) There is a computable copy of $\mathcal{F}_\infty$
  with no computable left order.

  (ii) Suppose $\mathcal{F}$ is a computable copy of $\mathcal{F}_\infty$,
  and let P be an order on $\mathcal{F}$.
  Suppose B is a basis for $\mathcal{F}$.
  Then for any $X >_T P \oplus B$,
  there is an order Q on $\mathcal{F}_\infty$ such that $Q \equiv_T X$.

  (iii) There is a computable copy of $\mathcal{F}_\infty$
  with a computable order and no c.e. basis.

- *Turing degree spectrum* of left-orders on computable G :

$$DgSp_G(LO) = \{\mathsf{deg}(P) \mid P \in LO(G)\}$$

$\mathsf{deg}(P) = \mathsf{deg}(\leq_P)$

$\mathcal{D}$ = the set of all Turing degrees

- (Solomon, 2002)

(i) $DgSp_G(LO) = \mathcal{D}$
for a torsion-free abelian group G of finite rank $n > 1$.

(ii) $DgSp_G(LO) \supseteq \{\mathbf{x} \in \mathcal{D} \mid \mathbf{x} \geq \mathbf{0'}\}$
for a torsion-free abelian group G of infinite rank.

(iii) $DgSp_G(LO) \supseteq \{\mathbf{x} \in \mathcal{D} \mid \mathbf{x} \geq \mathbf{0^{(n)}}\}$
for a torsion-free properly n-step nilpotent group G.

- A group G for which every partial (left) order can be extended to a total (left) order is called *fully orderable* (*fully left-orderable*).

  Torsion-free abelian groups are fully orderable.

- (Dabkowska, Dabkowski, Harizanov, Togha, 2010)
  Let G be a computable, *fully* left-orderable group and
  $\mathbf{d}$ a Turing degree such that:
  (a) No left order on G is determined uniquely by any finite subset of $G \backslash \{e\}$;
  (b) For a finite $A \subset G \backslash \{e\}$, the problem '$e \in sgr(A)$' is $\mathbf{d}$-decidable;
  (c) $DgSp_G(LO)$ closed upward.
  Then

  $$DgSp_G(LO) \supseteq \{\mathbf{a} \in \mathcal{D} \mid \mathbf{a} \geq \mathbf{d}\}$$

  and $LO(G)$ is homeomorphic to the *Cantor set*.

- Free group $F_n = \langle x_0, x_1, \ldots, x_{n-1} \mid \ \rangle$ of rank $n > 1$ is not fully left-orderable.

- (Dabkowska, Dabkowski, Harizanov, Togha, 2010)

  Let G be a computable group, $\mathbf{d}$ a Turing degree, $\mathbb{P} = \{p_i\}_{i \in \omega}$ a $\mathbf{d}$-computable strong array of finite subsets of $G \backslash \{e\}$ such that for every $p \in \mathbb{P}$, we have $e \notin sgr(p)$ and

  (a) there are $a \in G \backslash \{e\}$ and $q, r \in \mathbb{P}$ such that $q \supseteq p \wedge r \supseteq p$ and $a \in q \wedge a^{-1} \in r$;

  (b) for each $a \in G \backslash \{e\}$ there is $q \in \mathbb{P}$ such that $q \supseteq p$ and $a \in q \vee a^{-1} \in q$.

  Then $(\forall \mathbf{x} \geq \mathbf{d})(\exists \mathbf{z} \in DgSp_G(LO))[\mathbf{x} = \mathbf{z} \vee \mathbf{d}]$.

- *Corollary.* If $DgSp_G(LO)$ is closed upward, then $\{\mathbf{x} \in \mathcal{D} \mid \mathbf{x} \geq \mathbf{d}\} \subseteq DgSp_G(LO)$.

- (Dabkowska, Dabkowski, Harizanov, Togha, 2010) For the free group F_n of rank $n > 1$, we have $DgSp_{F_n}(BiO) = \mathcal{D}$.

  Proof idea:

  For a group G, the *lower central series* is the descending sequence of subgroups $\{\gamma_\alpha(G)\}$ defined as:

$$\begin{aligned} &\gamma_0(G) = G, \\ &\gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G], \\ &\gamma_\beta(G) = \textstyle\bigcap_{\alpha<\beta} \gamma_\alpha(G), \text{ when } \beta \text{ is a limit ordinal,} \end{aligned}$$

  where $[A, B]$ is the subgroup of G generated by the elements $a^{-1}b^{-1}ab$, with $a \in A$ and $b \in B$.

- *Lower central series* of F_n: $\gamma_1(F_n) \geq \cdots \geq \gamma_i(F_n) \geq \cdots$

- (Magnus) $\bigcap_{i=1}^{\infty} \gamma_i(F_n) = \{e\}$

- (Hall) $\gamma_i(F_n)/\gamma_{i+1}(F_n) \cong \mathbb{Z}^{k_i}$,
  where $k_i = \frac{1}{i}\sum_{d|i} \mu(\frac{i}{d})n^d$, μ Möbius function

- Isomorphism uniformly computable since a basis of $\gamma_i(F_n)/\gamma_{i+1}(F_n)$ can be found algorithmically in n, i.

- Construct bi-orders on F_n using bi-orders on $\gamma_i(F_n)/\gamma_{i+1}(F_n)$.

- Different choices of orders on quotients induce different orders on F_n.

- Produce a bi-order on F_n of a given Turing degree.

- A set A is *weak truth-table reducible* to a set B: $A \leq_{wtt} B$ if there is a computable function h and an index e so that

$$A(x) = \varphi_e^{B \upharpoonright h(x)}(x).$$

- tt-reducibility, a stronger notion, is a further refinement:

  A is *truth-table reducible* to B: $A \leq_{tt} B$
  if $A \leq_{wtt} B$ via φ_e^B and a computable function h,
  having the additional robustness property:
  for any string $\sigma \in 2^{<\omega}$ of length $h(x)$, $\varphi^\sigma(x) \downarrow$.

- (Chubb, Dabkowski, Harizanov, 2011)
  Let G be a group, and $\mathbb{P}$ a c.e. family of finite subsets of $G - \{e\}$ satisfying the following conditions for every $p \in \mathbb{P}$:

  (*purity*) $e \notin sgr(p)$;

  (*branching*) $(\exists q, r \in \mathbb{P})(\exists a \in G)[q, r \supseteq p \wedge a \in q \wedge a^{-1} \in r]$;

  (*extendability*) $(\forall a \in G - \{e\})(\exists q \in \mathbb{P})[q \supseteq p \wedge (a \in q \vee a^{-1} \in q)]$.

  Then G admits a bi-order in each tt-degree.

- We now generalize the construction for free groups of finite rank > 1 to a class of finitely presented, residually nilpotent groups that are not nilpotent.

- (Chubb, Dabkowski, Harizanov, 2011)
  Let G be a finitely-presented, torsion-free, computable group.
  Let $G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots$ be the lower central series of G.
  If $\gamma_\omega(G) = \{e\}$ and $\gamma_i(G)/\gamma_{i+1}(G)$ is non-trivial and torsion-free for each $i = 1, 2, \ldots,$ then there is a bi-order on G in every tt-degree.

- Conjecture (Sikora, 2004)
  The space $BiO(F_n)$ for $n > 1$ is homeomorphic to the Cantor set.

- (Navas-Flores, 2008)
  The space $LO(F_n)$ for $n > 1$ is homeomorphic to the Cantor set.

THANK YOU!