Jump classes and automorphisms of the c.e. sets

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Computably enumerable sets

Definition

A set A is *computably enumerable* (c.e.) if it is the domain W_e of a partial computable function Φ_e.

Equivalently:

- A is Σ_1^0.
- There is a computable listing of the elements in A.
- A is the range of a computable function (or empty).
- $A \leq_1 K$, where K is the halting set $\{e : e \in W_e\}$.

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Low$= L_1 = \{\mathbf{d} \mid \mathbf{d}' = \mathbf{0}'\}$.

High $= H_1 = \{\mathbf{d} \mid \mathbf{d}' = \mathbf{0}''\}$.

Definition

Low$_n = L_n = \{\mathbf{d} \mid \mathbf{d}^{(\mathbf{n})} = \mathbf{0}^{(\mathbf{n})}\}$

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- Sacks proved the Jump Inversion Theorem, which led to the following corollary:

Corollary

$\mathbf{0} = L_0 \subsetneq L_1 \subsetneq L_2 \subsetneq L_3 \subsetneq \ldots$, and

$\mathbf{0}' = H_0 \subsetneq H_1 \subsetneq H_2 \subsetneq H_3 \ldots$ within the c.e. degrees.

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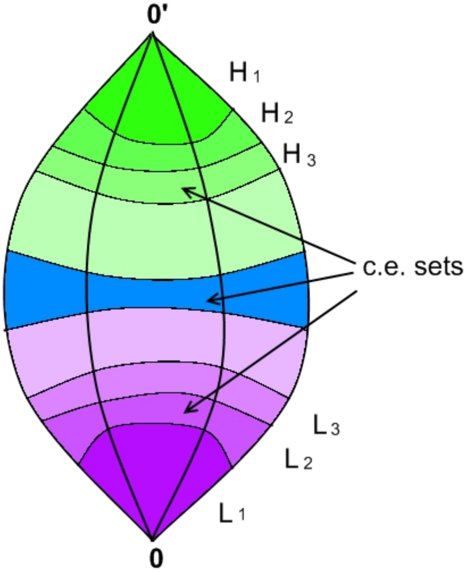
0'
H 1
H 2
H 3
c.e. sets
L 3
L 2
L 1
0

Definition

Let $\mathcal{E}$ be the lattice of the c.e. sets: $\mathcal{E} = \{\{W_e\}_{e\in\omega}, \cup, \cap, \omega, \emptyset\}$.

- A class of c.e. sets is definable in $\mathcal{E}$ if it can be defined in the language of set inclusion.
- Computable sets $=$ complemented sets.
- Finite sets $F = \{W \in \mathcal{E} \mid (\forall X \subset W)[X \text{ is computable}]\}$.
- Let $\mathcal{E}^* = \mathcal{E}/F$.
  For our purposes, anything we want to say about $\mathcal{E}$, we can prove about $\mathcal{E}^*$ instead.

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We say a class of degrees $\mathcal{C}$ is *definable* if $\mathcal{C} = \{\deg(W) \mid W \in \mathcal{S}\}$ where $\mathcal{S}$ is a class of sets definable in $\mathcal{E}$.

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Which classes of degrees are definable in $\mathcal{E}$?

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Which jump classes (L_n, H_n, $\overline{L_n}$, $\overline{H_n}$) are definable in $\mathcal{E}$?

(It suffices to show which are definable in $\mathcal{E}^*$ [Lachlan])

- $L_0 = \{\mathbf{0}\}$: Definable by $\{\deg(\emptyset)\}$.
- $\overline{L_0} = \{\mathbf{d} \mid \mathbf{d} > \mathbf{0}\}$: Definable by $\{\deg(W) \mid \overline{W} \notin \mathcal{E}\}$.

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$$A \subset W_e \quad \implies \quad W_e =^* A \quad \text{or} \quad W_e =^* \omega.$$

Theorem (Martin, 1966)

H_1 = the degrees of maximal sets.

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A set A is *atomless* if it is not contained in any maximal set.

- Lachlan [1968]: The atomless sets are contained in the class $\overline{L_2}$.
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Known results in 1986

Red = Definable Blue = Not definable

- $L_0 = \{\mathbf{0}\}$
- $\overline{L_0} = \{\mathbf{d} \mid \mathbf{d} > \mathbf{0}\}$
- $H_0 = \{\mathbf{0}'\}$: Definable because the creative sets are definable [Harrington, 1986].
- $H_1 = \{\mathbf{d} \mid \mathbf{d}' = \mathbf{0}''\}$ by Martin
- $\overline{L_2} = \{\mathbf{d} \mid \mathbf{d}'' > \mathbf{0}''\}$ by Lachlan-Shoenfield

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A class of sets $\mathcal{S} \subseteq \mathcal{E}$ is *invariant* if it is closed under $\mathrm{Aut}(\mathcal{E})$. A class of degrees $\mathcal{C}$ is *invariant* if $\mathcal{C} = \{\deg(W) \mid W \in \mathcal{S}\}$, where $\mathcal{S}$ is invariant.

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Theorem (Cholak 1995, Harrington-Soare 1996)

Every noncomputable c.e. set is automorphic to a high set.

Corollary

All downward closed jump classes L_n, $\overline{H_n}$, $n \geq 1$, are noninvariant, and thus not definable.

Theorem (Harrington-Soare, 1996)

For all prompt sets A, there exists $B \equiv_{\mathrm{T}} \mathbf{0}'$ such that $A \simeq B$.

Corollary

$\overline{H_0}$, the degrees below $\mathbf{0}'$, are not definable.

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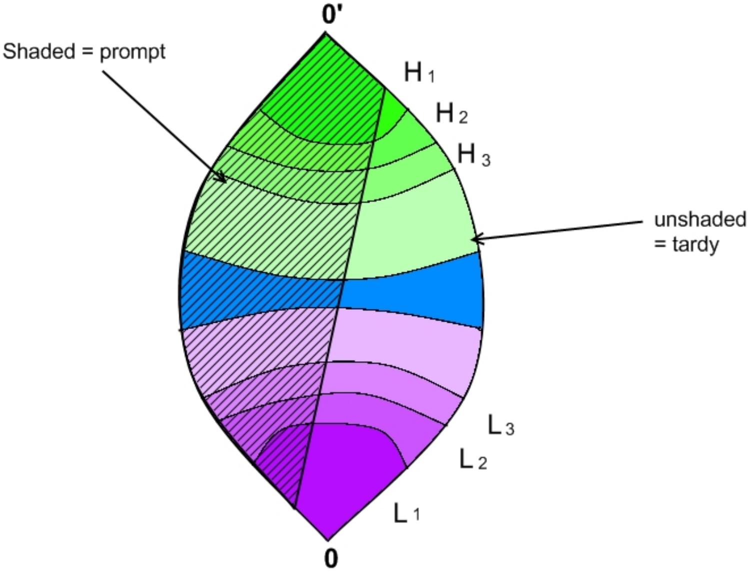
0'
Shaded = prompt
H 1
H 2
H 3
unshaded
= tardy
L 3
L 2
L 1
0

The situation in 1996

Red = Definable Blue = Not definable

Upward Closed		Downward Closed	
nonlow$_n$	high$_n$	low$_n$	nonhigh$_n$
$\overline{L_0}$	H_0	L_0	$\overline{H_0}$
$\overline{L_1}$	H_1	L_1	$\overline{H_1}$
$\overline{L_2}$	H_2	L_2	$\overline{H_2}$
$\overline{L_3}$	H_3	L_3	$\overline{H_3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The situation in 2002

Theorem (Cholak-Harrington, 2002)

For $n \geq 2$, H_n and $\overline{L_n}$ are definable.

Red = Definable   Blue = Not definable

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Conjecture (Harrington-Soare, 1996)

$\overline{L_1}$ *is noninvariant.*

Theorem (Epstein)

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Theorem (Epstein)

There exists a nonlow D such that for all $A \leq_T D$, there exists a low set B such that $A \simeq B$.

Corollary (Epstein)

The nonlow degrees are noninvariant, and thus not definable.

Proof: Let $\mathbf{d} = \deg(D)$. Then $\mathbf{d}$ is an $\overline{L_1}$ degree such that all sets in $\mathbf{d}$ are automorphic to low sets.

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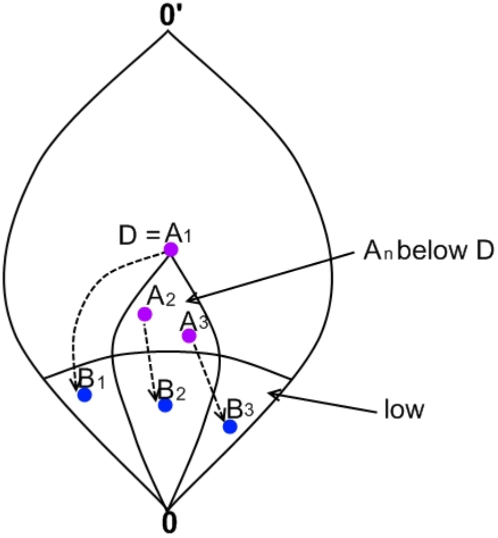
0'
D = A1
An below D
A2
A3
B1
B2
B3
low
0

Building an automorphism

Red = Things we are given Blue = Things we build

- Given an enumeration $\{U_n\}_{n\in\omega}$ of the c.e. sets, where $U_0 = A$.
- Build an enumeration $\{\widehat{U_n}\}_{n\in\omega}$ of the c.e. sets. Let $B = \widehat{U_0}$.
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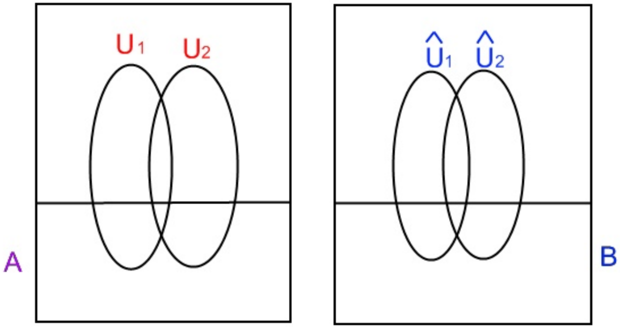
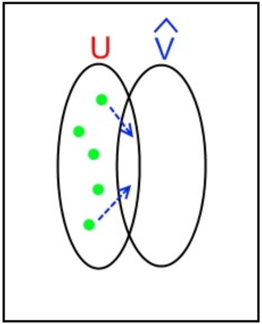
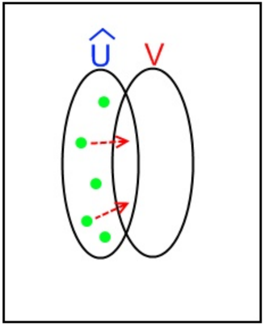
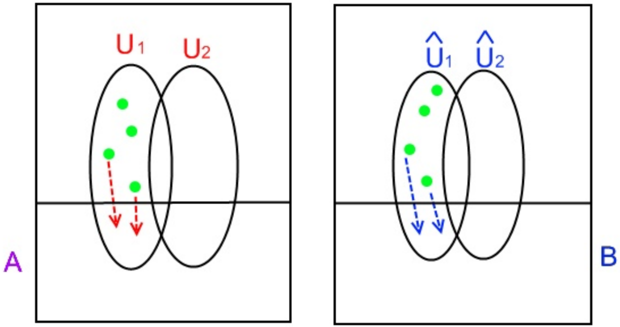

U_1
U_2
A
$\hat{U}_1$
$\hat{U}_2$
B



U
V̂



Û
V


U1
U2
A
Û1
Û2
B

Recall:

Theorem (Harrington-Soare, 1996)

For all prompt sets A, there exists $B \equiv_T \mathbf{0}'$ such that $A \simeq B$.

Theorem (Cholak 1995, Harrington-Soare 1996)

For all noncomputable A, there exists B high such that $A \simeq B$.

These theorems move sets *up* in degree. We move sets *down*.

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- It does not allow us to restrain elements from falling into *A*.

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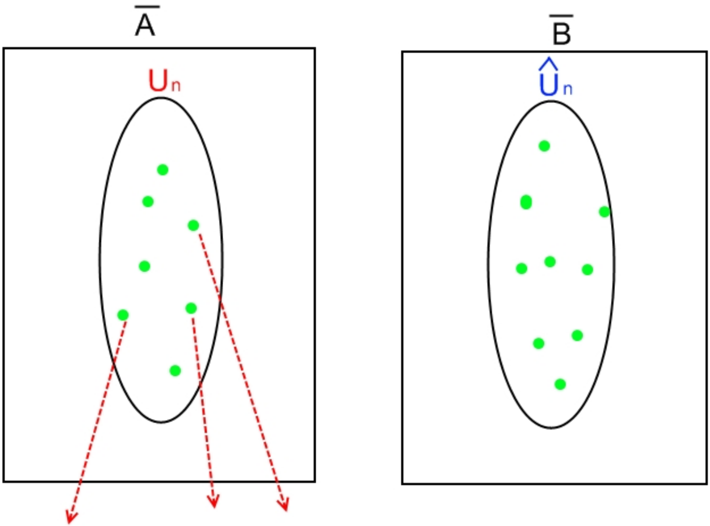
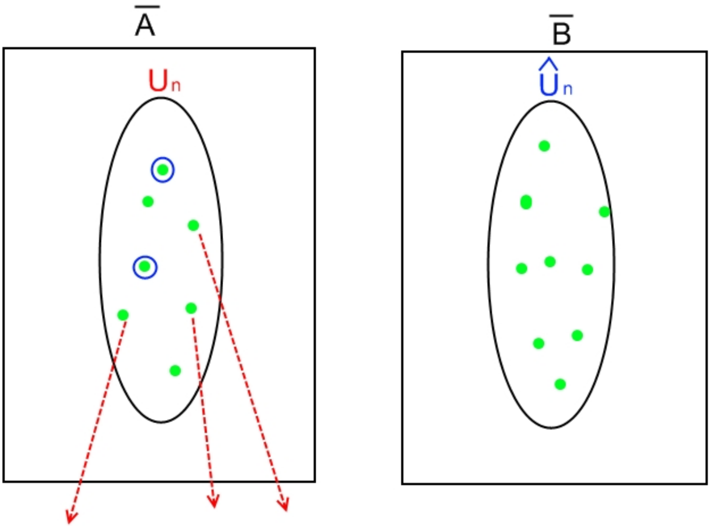


$\overline{A}$
U_n
$\overline{B}$
$\hat{U}_n$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Other automorphism questions

This proves that there is a nonlow degree whose lower cone contains only sets automorphic to low sets.

Question

What else can we say about sets automorphic to low sets?

Theorem (Harrington-Soare, 1998)

There is a low$_2$ promptly simple set A with $\overline{A}$ semi-low$_{1.5}$ such that A is not automorphic to a low set.

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- We usually build Δ^0_3 automorphisms of $\mathcal{E}^*$
- Which sets are Δ^0_3-automorphic to low sets?

Conjecture (Cholak-Weber)

The sets with the Δ^0_3-low shrinking property are precisely the sets Δ^0_3-automorphic to low sets.

Definition (Maass, 1985)

A has the (Δ^0_3)*low shrinking property* if for any enumeration $\{W_e\}$ of the c.e. sets, there is a computable (Δ^0_3) function f such that

$$W_{f(e)} \subseteq W_e \quad \& \quad W_{f(e)} \cap \overline{A} =^* W_e \cap \overline{A},$$

and for all finite $I \subset \omega$

$$A \cap \bigcap_{i \in I} W_{f(i)} \text{ infinite} \implies \overline{A} \cap \bigcap_{i \in I} W_{f(i)} \textit{infinite}.$$

- Which sets are effectively automorphic to low sets?
- Soare [1982] showed that all sets with semilow complement are effectively automorphic to low sets, where S is semilow if $\{e : W_e \cap S \neq \emptyset\} \leq_T \mathbf{0}'$.
- Two conjectures:
  - A is effectively automorphic to a low set if $\overline{A}$ is semilow.
  - A is effectively automorphic to a low set if A has the low shrinking property.
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Which sets are automorphic to complete sets?

- All prompt sets are Δ_3^0-automorphic to complete sets (Harrington-Soare, 1996).
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Theorem (Harrington)

For all noncomputable c.e. sets A and all c.e. $C <_{\mathrm{T}} \mathbf{0}'$, there is a c.e. set $B \not\leq_{\mathrm{T}} C$ such that $A \simeq B$.

Question (Avoiding an upper cone)

For all c.e. sets $A < \mathbf{0}'$ and noncomputable c.e. sets C, is there a c.e. set B, $C \not\leq_{\mathrm{T}} B$, such that $A \simeq B$?

Theorem (R. Miller, 2002)

True for A low.

Corollary (Epstein, R. Miller)

There exists a nonlow c.e. set D such that for all $A \leq_{\mathrm{T}} D$ and all $C >_{T} \mathbf{0}$, there is a c.e. set B, $C \not\leq_{\mathrm{T}} B$ and $A \simeq B$.

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Thanks for listening!