A Computational Approach to an Alternative Working Environment for the Constructible Hierarchy

Merlin Carl

Infinitary machine models

Infinitary machine models generalize classical concepts of computability to the infinite by extending the "running time" and, possibly, the memory.

The first and most famous example are Infinite Time Turing Machines (ITTM's), introduced by Hamkins and Lewis.
These can be imagined as Turing machines with some limit ordinals α and β as tape length and running time, respectively.

Another example are the Ordinal Register Machines (ORM's) by Koepke and Miller.

Ordinal Register Machines

Ordinal register machines (ORM's) generalize classical register machines to computations on ordinals.

Definition: An ORM program is a finite list $P = (P_0, P_1, ..., P_k)$ of instructions acting of registers $(R_i | i \in \omega)$. The index i of the instruction P_i is also called the state of P_i. An instruction may be of one of four kinds:

a) the zero instruction $Z(n)$ changes the contents of R_n to 0, leaving all other registers unaltered;

b) the successor instruction $S(n)$ increases the ordinal contained in R_n, leaving all other registers unaltered;

c) the transfer instruction $T(m, n)$ sets the contents of R_n to the contents of R_m, leaving all other registers unaltered;

d) the jump instruction $P_i = J(m, n, q)$ is carried out within the program P as follows: the contents r_m and r_n of the registers R_m and R_n are compared, all registers are left unaltered; then, if $r_m = r_n$, the ORM proceeds to the instruction P_q of P; if $r_m \neq r_n$, the ORM proceeds to the next instruction P_{i+1} in P.

Computations according to a program P are now carried out along an ordinal time axis, where the successor instructions are carried out classically while at limit steps, the new machine state is obtained by taking the limes inferior of the earlier states.

ORMs and L

Say a set x of ordinals is ORM-computable if there is an ORM-program P and a finite set of ordinals p such that P computes the characteristic function χ_x of x with initial input p. The computational power of these machines is then given by the following theorem:

Theorem: x is ORM-computable iff $x \in L$, where L is Gödel's constructible universe. (see below)

Definable sets

Let $LAST$ be the usual language of set theory, i.e. a first-order language with equality containing a single binary relation symbol ϵ. Fix $(\phi_i | i \in \omega)$, an effective enumeration of the $LAST$-formulas in order type ω.

Then, if X is a set, ϕ a $LAST$-formula, $\vec{y}$ a finite vector from X, we set:

$I(X, \phi, \vec{y}) := \{x \in X | \phi^X(x, \vec{y})\}$.

I is called the intepretation function. Sets of the form $I(X, \phi, \vec{y})$ are called definable subsets of X.

V and L

$V_0 := \emptyset$

$V_{\alpha+1} := \mathbf{P}(V_\alpha)$

$V_\lambda := \bigcup_{\iota<\lambda} V_\iota$ for λ a limit ordinal

Finally, $V := \bigcup_{\iota \in On} V_\iota$.

Now, we let:

$L_0 := \emptyset$

$L_{\alpha+1} := \{I(L_\alpha, \phi, \vec{y}) | \phi$ a $LAST$-formula, $\vec{y}$ a finite vector from $L_\alpha\}$.

$L_\lambda := \bigcup_{\iota<\lambda} L_\iota$ for λ a limit ordinal

Then, let $L := \bigcup_{\iota \in On} L_\iota$.

L is called the constructible universe.

Properties of L

- $\alpha < \beta$ implies $L_\alpha \subset L_\beta$ and $L_\alpha \in L_\beta$.

- L_α is transitive for every α.

- L is the $\subseteq$-minimal transitive class model of ZFC

- L is absolute between transitive class models of ZFC, i.e. $L^M = L^V$ for every transitive class model M of ZFC. In particular, $L^L = L$.

- The statement $V = L$, i.e. that every set is constructible, is expressible in $LAST$ and consistent relative to ZFC.

- L has a Σ_1-definable well-ordering $<_L$.

Condensation

A very convenient property of L is given by Gödels condensation lemma:

If $X \prec_{\Sigma_1} L$, then either $X \cong L$ or there is $\alpha \in On$ such that $X \cong L_\alpha$.

The isomorphism is in each case just the Mostowski collapse.

CH in L

Let $\omega \supset x \in L$.

Consider $H := \Sigma_1^L\{\omega \cup \{x\}\}$.

Let $L_\alpha \cong H$ by the condensation lemma, and let π be the collapsing map. Then $\pi|\omega = id$, so $x = \pi(x) \in L_\alpha$.

Since H is countable, so is L_α and hence α.

So all subsets of ω in L are already elements of L_{ω_1}. But it is easy to see that $|L_{\omega_1}| = \omega_1$ in L. Hence CH holds in L.

Constructibility Theory

For stronger applications of L, a deeper understanding of the combinatorial structure of L is required. There are some strong principles, due to Jensen, which are commonly used in constructible combinatorics. One of the most famous examples is Jensens $\Box$ principle:

There is a sequence $(C_\beta | \beta$ a singular ordinal) such that:

(1) C_β is closed and unbounded in β

(2) $otp(C_\beta) < \beta$

(3) If $\bar{\beta}$ is a limit point of C_β, then $\bar{\beta}$ is singular and $C_{\bar{\beta}} = C_\beta \cap \bar{\beta}$.

A $\Box$-sequence is hence a uniform simultaneous singularization for L. This principle is e.g. used to show that the generalized Souslin hypothesis fails at all successor cardinals in L.

Technicalities

The main tools of constructibility theory employed to prove such results are definable Skolem functions, the comprehension operator (our interpretation function I), naming (a kind of an inverse function for I) and the well-ordering $<_L$.

A canonical Skolem function $h(\phi, \vec{y})$ for L_α gives the $<_L$-minimal witness w in L_α for $\phi(w, \vec{y})$ if it exists and is otherwise undefined.

In the original approach, these tools have to be obtained with considerable technical effort.

Fine structure theory

The detailed analysis of how and when new sets arise in the construction of the L-hierarchy is referred to as 'fine structure theory'. It originates in Jensen's celebrated work 'On the fine structure of the constructible hierarchy'.

Fine structure theory is necessary for proofs of strong statements in L: E.g. for proving $\Box$ in L, one has to consider for each singular ordinal β the lexically minimal triple $(\alpha(\beta), n(\beta), p(\beta))$ such that β is singularized by a function f that is $\Sigma_{n(\beta)}$-definable over $L_{\alpha(\beta)}$ in parameter $p(\beta)$.

Fine structure theory

Typically in fine-structural proofs, the Σ_1 case (i.e. when $n(\beta) = 1$) is easily handled and contains the combinatorial core of the argument.

For the other cases, the main idea is to consider L-levels relativized to recursive truth predicates in such a way that a Σ_n-definition can be viewed as a Σ_1-definition over the relativized structure.

This reduction procedure is technically rather involved and is typically responsible for at least half the length of the proof.

Important operators

In finestructural analysis, the following operators and relations occur frequently:

- The constructible well-ordering $<_L$
- The interpretation operator $I(X, \phi, \vec{z}) = \{x \in X | \phi^X(x, \vec{z})\}$
- The naming operator N, obtaining for each $x \in L$ a lexically minimal inverse under the interpretation function
- Bounded Skolem functions $S(X, \phi, \vec{z})$, finding a $<_L$-minimal witness $y \in X$ for $\phi^X(y, \vec{z})$

In classical fine structure, these have to be obtained with considerable effort.

These operators correspond to computability concepts, e.g.:

- Since the bounded truth predicate over L is ORM-computable, the interpretation function corresponds to a decision procedure for the set obtained

- The Skolem function is analogous to the μ-operator from classical computability theory

Simplifications and Machine-based fine structure

After the development of fine structure theory, several alternatives were proposed to simplify arguments or at least to make them more intuitive.

The most prominent of these attempts are 'Silver machines', very general sequences of hull operators of slowly increasing strength. This approach avoided many cumbersome technicalities of classical fine structure.

The arguments using Silver machines can be modified to fit into the framework of ORM's, leading to proofs concerning programs rather than structures.

However, directly using a particular machine model obscures the original intuition behind the constructible hierarchy. A way to combine the advantages of both approaches would be to consider a hierarchy of levels equipped with the necessary operations, but without reference to programs or a particular machine model.

A 'higher programming language' for finestructural analysis

Based on these considerations, a natural attempt at a simplified finestructure would be to integrate these basic functions directly into the underlying language of set theory.
We should expect two advantages:

1) The technical effort for constructing the basic functions in the $\in$-language is reduced

2) More importantly, since the augmented language contains the relevant operators as primitive symbols, we can work with a restricted fragment of it, e.g. quantifier-free formulas.

The F-hierarchy

This is the idea of the 'fine hierarchy', a relatively new approach to simplified fine structure, originally due to Koepke and van Eijmeren: Build the operators and relations necessary for finestructural analysis directly into the underlying language and set up the hierarchy in such a way that sets appearing at a certain stage have very simple definitions in the modified language.

Recall that the main tools we singled out so far were the comprehension operator I, the Skolem function S, a naming function N and a well-ordering $<_L$.

The language **S**

Let **S** be the first-order language with countably many variable symbols, the usual logical connectives and quantifiers amended with function symbols I and S of variable finite arity, a binary relation symbol $\in$ and a ternary relation symbol $x <_y z$.
S-terms and formulas are defined in the obvious way. By S_0, we denote the set of quantifier-free formulas of **S**.

The F-hierarchy

The F-hierarchy is now obtained by iterating S_0-definability.

$\mathbf{F}_\alpha = (F_\alpha, I_\alpha, N_\alpha, S_\alpha, <_\alpha, \in)$, defined recursively as follows:

For $\alpha \leq \omega$, we let $F_\alpha := V_\alpha$, $I_\alpha = S_\alpha = const_0$. $<_\omega$ is an arbitrary recursive well-order of F_ω extending $\in$.
Now, let $F_{\alpha+1} := \{I(F_\alpha, \phi, \vec{p}) | \phi \in S_0$ and $\vec{p} \subset_{fin} F_\alpha\}$.
The definition of I is now clear. $N(x)$ denotes the lexically $<_\alpha <_{\alpha+1}$-smallest triple (a, b, c) such that $I(a, b, c) = x$.
$<_{\alpha+1}$ is defined to extend $<_\alpha$ such that, for $x, y \in F_{\alpha+1} - F_\alpha$, we let $x <_{\alpha+1} y$ if $N(x)$ is lexically smaller than $N(y)$.
$S_{\alpha+1}(\phi, \vec{p})$ is now the $<_{\alpha+1}$-smallest witness w for $\phi(x, \vec{p})$ if such w exists, and otherwise $\emptyset$.

The F-hierarchy - continued

If λ is a limit ordinal bigger than ω, then we let:

- $F_\lambda := \bigcup_{\iota<\lambda} F_\iota$
- $S_\lambda := \bigcup_{\iota<\lambda} S_\iota$
- $N_\lambda := \bigcup_{\iota<\lambda} N_\iota$
- $I_\lambda := \bigcup_{\iota<\lambda} I_\iota$
- $<_\lambda := \bigcup_{\iota<\lambda} <_\iota$

It is easy to see that the functions and relations agree on their common domains, so that this definition works.

F and J

The hierarchy used in most finestructural proof is Jensen's J-hierarchy, a convenient modification of the L-hierarchy. By a result of Koepke and van Eijmeren, it relates to the F-hierarchy in a simple way:

$J_\alpha = F_{\omega \cdot \alpha}$ for every $\alpha \in On$.

Hence, the F-hierarchy is a refinement of the J-hierarchy. This already indicates that it is a suitable framework for finestructural arguments.

Properties of the F-hierarchy

- For each α, F_α is transitive
- $\alpha < \beta$ implies $F_\alpha \subset F_\beta$ and $F_\alpha \in F_\beta$
- $\bigcup_{\iota \in On} F_\iota = L$
- Condensation: If $X \subset L$ is closed under I, N, and S, then $X \cong L$ or $X \cong F_\alpha$ for some $\alpha \in On$.
- Emulation of first-order-definability: If ϕ is a $LAST$-formula, there exist (effectively) k and $\phi^* \in S_0$ such that $(F_\alpha, \in) \models \phi(x)$ iff $F_{\alpha+k} \models \phi^*(x, F_\alpha, ..., F_{\alpha+k})$.

Hence, the F-hierarchy satisfies the same crucial properties as the L-hierarchy or Jensen's J-hierarchy. But in addition, every set in L now has a quantifier-free definition over a level of the hierarchy.

Coding embeddings

We consider some more algebraic notions related to the F-hierarchy.

A function $\sigma : F_\alpha \to F_\beta$ is fine iff $\pi | F_\omega = id$ and σ preserves S_0-formulas.

If we additionally have that $I(F_\beta, \phi, \vec{z}) \in rng(\sigma)$ and $S(F_\beta, \phi, \vec{z}) \in rng(\sigma)$ whenever $I(F_\beta, \phi, \vec{z}) \in F_\beta$, $S(F_\beta, \phi, \vec{z}) \in F_\beta$, respectively, then σ is called 'fine up to F_β'.

Also, if $X \subset L$, then we denote by $F\{X\}$ the fine hull of X, i.e. the closure of X under I, N and S.

Fact: If σ is fine up to F_β, then there is a fine $\sigma^+ : F_{\alpha+1} \to F_{\beta+1}$ such that $\sigma \subset \sigma^+$.

Fine systems

Let $(P,<)$ be a directed poset, $T := (F_{\alpha_i}, \pi_{ij} | i,j \in P, i < j)$ such that $\pi_{ij} : F_{\alpha_i} \to F_{\alpha_j}$ is fine. Then T is called a fine system.

Fact 1: A fine system T has a unique direct limit A^T, together with fine maps $\pi_i : F_{\alpha_i} \to A^T$ for every $i \in P$.

Fact 2: If A^T is well-founded, then there is γ such that $A^T \simeq F_\gamma$

Extensions of embeddings I

F_α is a base for F_η iff, for every $\mu < \alpha$, $p \subset_{fin} F_\eta$, we have $F\{F_\mu \cup p\} \simeq F_{\gamma(\mu,p)}$ with $\gamma(\mu,p) < \alpha$.

For α a limit ordinal and a base for F_γ, consider the fine system $T_\alpha^\gamma := (F_{\gamma(\mu,p)}, \pi_{\mu,p,\mu',p'} | (\mu \leq \mu' < \alpha, p \subseteq p' \subset_{fin} F_\gamma)$.

Fact 1: $\pi_{\mu,p,\mu',p'} \in F_\alpha$.

Fact 2: $dirlim(T_\alpha^\gamma) = F_\gamma$

Extensions of embeddings II

Suppose $E : F_\alpha \to F_\beta$ is fine, and that F_α is a base for F_γ. Since the components of T^γ_α are elements of F_α, we can form $E[T^\gamma_\alpha]$.

$E[T^\gamma_\alpha]$ is easily seen to be a fine system. Let $A := dirlim(T^\gamma_\alpha)$. We denote this A by $Ext(F_\gamma, E)$. Assuming that A is well-founded, we have $A \simeq F_\delta$ for some $\delta \in On$.

We can then define the extension map $\pi^E : F_\gamma \to F_\delta$ as follows:

Suppose $x \in rng(\pi_i)$. Then let $\pi^E(x) := \pi'_{E(i)} \circ E \circ \pi_i^{-1}(x)$.

Fact: If the direct limit A is well-founded with transitive collapse F_δ, then $\pi^E : F_\gamma \to F_\delta$ is fine up to F_δ.

A sample application

We consider a variant of a theorem of Magidor:

Assume that there is no fine $\sigma : L \to L$.
Let $X \subset On$ be such that $F\{X\} \cap On = X$.
Then there are $(x_i | i \in \omega)$ such that $x_i \in L$ and $X = \bigcup_{i \in \omega} x_i$.

Preliminaries

A property stronger than being a base is being an ω-base:

F_α is an ω-base for F_γ iff, for $\bar{\alpha} < \alpha$ and a countable $A \subset F_\gamma$, we have $F\{F_{\bar{\alpha}} \cup A\} \simeq F_{\alpha(\bar{\alpha},A)}$ with $\alpha(\bar{\alpha}, A) < \alpha$.

Fact: If $E : F_\alpha \to_{fine} F_\beta$ and F_α is an ω-base for F_γ, then $Ext(F_\gamma, E)$ is well-founded (and hence isomorphic to some fine level).

Proof sketch
We proceed by induction on $\beta := sup(X)$.
For $\beta = 0$, there is nothing to show.
The cases where β is a successor or a limit of cofinality ω, the induction step is trivial.
So let wlog $cf(\beta) > \omega$.

Form $F\{X\}$, and let $E : F_\alpha \to F\{X\}$ be the collapsing map. Obviously, we then have $rng(E) \cap \beta = X$.We may wlog assume that $E \neq id$.

Observation: F_α cannot be an ω-base for L, since otherwise, E could be extended to a non-trivial fine $\pi : L \to L$; contradicting our assumption.

Proof sketch III

Let ρ be minimal such that F_α is not an ω-base for F_ρ, and pick $\bar{\alpha} < \alpha$, $A \subset F_\rho$ countable such that $F\{F_{\bar{\alpha}} \cup A\} \simeq F_\zeta$ with $\zeta \geq \alpha$.

Wlog let $A = coll[A]$. Then $F_\zeta = F\{F_\rho \cup A\}$ and, by minimality of ρ, hence $\zeta = \rho$.

Fact: Either ρ is a successor or $cf(\rho) = \omega$. (Since $A = \cup_{i\in\omega} A_i$, where A_i is a proper initial segment of A_j for $i < j$ and further $sup(On \cap F\{F_{\alpha(\rho)} \cup A_i\} \leq sup(A_i \cap On) + 1$.)

We now distinguish two cases, according to whether ρ is a successor or a limit.

The successor case

Let $\rho = \gamma + 1$. So F_α is an ω-base for F_γ, hence there is

$E \subset \pi : F_\gamma \to F_\delta$.

Obviously $E[\alpha] = \pi_E[\alpha] = \pi_E[F_\gamma] \cap \beta = X$.

Since π_E is fine up to F_δ, there is

$\pi_E \subset \pi_E^+ : F_{\gamma+1} \to F_{\delta+1}$.

Hence: $X = \beta \cap rng(\pi_E^+) = \beta \cap \pi_E^{+''} F_\rho = \beta \cap \pi_E^{+''} F\{F_{\bar\alpha} \cup A\}$

$= \beta \cap F\{\pi_E^{+''} F_{\bar\alpha} \cup \pi_E^{+''} A\} = \beta \cap F\{\pi_E^{+''} \underline{F}_{\bar\alpha} \cup \pi_E^{+''} A\}$

$= \beta \cap F\{E'' \underline{F}_{\bar\alpha} \cup \pi_E^{+''} A\}$.

Now we can use the induction hypothesis on $E[F_{\bar\alpha}]$ to obtain the result.

In the limit case, almost the same reasoning applies, with the difference that we simultanously consider the countably many directed systems $T_\alpha^{\rho_i}$, where $(\rho_i|i \in \omega)$ is cofinal in ρ.

This finishes the proof.

In contrast to the classical approach, no case distinction according to definitional complexity was necessary.
Due to the integration of the right 'macros' into the structure, the language could be kept 'flat'. This resulted in a considerable simplification of the proof.

However, problems arise in applications of a more combinatorial nature, like e.g. Jensen's square principle mentioned above.
This theorem is proved by chosing, for each singular α, a singularizing function f of imnimal definitional complexity. Here, it is crucial that f arises over a limit stage and can thus be approximated nicely from below.

In the F-hierarchy, however, such objects and effects can occur over successor stages.

A possible solution for this problem is to refine the hierarchy by adding suitable operators that interpolate between its levels.

For example, we can introduce Skolem functions with restricted domain and expand it in each step as little as possible.

This is the underlying intuition of hyperfine structure, first introduced and studied by Friedman and Koepke over the L-hierarchy.

Hyperings: Hyperfine structure over F-levels

Definition: A **location** is a countable sequence of the form $\langle F_\alpha, x_1, x_2, ...\rangle$, where each x_i is either a finite sequence of elements of $\underline{F_\alpha}$ or Ω. If the ordinal in the first position is β, it is a β-**location**. If $s = \langle F_\alpha, x_1, x_2, ...\rangle$ is a location, the set $supp(s) := \{i \in \omega | x_i \neq \vec{0}\}$ is called the **support** of s.

Let $s = \langle F_\alpha, x_1, ...\rangle$ be a location. The **structure corresponding to s** is defined thus:

$\langle F_\alpha, I, S, I|x_1,$

$N \circ I|x_1, S|x_1, N \circ S|x_1, I|x_2, N \circ I|x_2, S|x_2, N \circ S|x_2, ...\rangle$

For $X \subset F_\alpha$, the s-hull of X, written $F_s\{X\}$, is the closure of X under all functions belonging to s.

$<_{loc}$ denotes the lexical ordering (from the left) of locations.

A location s_2 is a successor of a location s_1 if $(s_1)_0 = (s_2)_0$ and further $(s_2)_i = (s_1)_i$ or $(s_2)_i$ is the immediate lexical successor of $(s_1)_i$ for each $i \geq 1$. Here, $(s)_i$ is the ith member of the sequence s.
(Note that, in contrast to classical hyperfine structure, these locations refer to F-levels. In this context, locations in the sense of Friedman and Koepke are a special case of the notion introduced here.)

Definition: Suppose A is a class of locations, $<_H \subset A \times A$. Then $H := \langle A, <_H \rangle$ is a linear hypering of L if the following axioms are satisfied:

1. (1) $<_H = <_{loc} |_H$ and the latter is a well-order

2. (2) For each location $l^n_\alpha(x)$, there is an α-location $s = \langle \alpha, y_1, y_2, ... \rangle$ in H such that $y_n = x$

3. (3) For each ordinal α, both $l_\alpha(\vec{0})$ and $l_\alpha(\Omega)$ are in H

4. (4) For $S \subset H$ a set of α-locations, $<_{loc}$-$sup\{S\} \in H$

5. (5) If $s_1, s_2 \in H$ are α-locations, H contains a (naturally unique) chain from s_1 to s_2

6. (6) If $\alpha < \beta$, $\pi : F_\alpha \rightarrow F_\beta$ a fine map, then $s \in H_\alpha$ iff $\pi(s) \in H_\beta$.

Future goals

One motivation for the introduction of the F-hierarchy was to obtain simplifications for the finestructure of core models as well. Core models are L-like structures that may contain certain large cardinals that cannot exist in L (e.g. measurable cardinals). For this, a suitable concept of a fine ultrapower has to be defined. In spite of several attempts, this is an open challenge so far.

Furthermore, it is also desirable to see what happens if the F-approach is applied to other combinatorial statements.

Thank you!