Minimality and Jump Classes

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We study the Turing degrees and the Turing reducibility $\leq$.

Question

What properties are definable in $\mathcal{L}_0 = \{\leq, \mathbf{0}\}$?

Fact

The only properties definable by quantifier-free $\mathcal{L}_0$ formulae (without other constants) are recursive and nonrecursive.

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Theorem [Jockusch, Shore]

A degree $\mathbf{x}$ is arithmetic ($\leq 0^{(n)}$ for some $n \in \omega$) if and only if there is a $\mathbf{y} \geq \mathbf{x}$ such that for any $\mathbf{z}$, $\mathbf{y} \vee \mathbf{z}$ is not a minimal cover of $\mathbf{z}$.

Fact

If $\varphi(x)$ is of the form $\exists y \psi(x, y)$ where ψ is quantifier-free, then we have the following (nontrivial) cases:

- y is strictly above x;
- y is incomparable with x;
- y is strictly below x and strictly above $\mathbf{0}$.

Definition

A nonrecursive degree $\mathbf{d}$ is minimal if the interval $(\mathbf{0}, \mathbf{d})$ is empty.

Proposition

The only properties definable by $\exists$ or $\forall$ formulae in $\mathcal{L}_0$ are recursive, nonrecursive, minimal and nonminimal.

Two quantifier case is much more complicated. With $\exists y \forall z \psi(x, y, z)$ one can write out a number of nontrivial facts:

Case 1

There is a y incomparable with x such that for every z strictly below x, z is below y.

Proposition

The above property is equivalent to x being join-irreducible, i.e., x is not the join of two degrees strictly below it.

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Case 2

There is a y strictly above x such that for every z strictly below y, z is below x.

Definition

In the above case, y is called a strong minimal cover of x; if we only require that (x, y) is empty, then y is called a minimal cover of x.

Theorem [Spector]

Every degree has a minimal cover.

Case 3

There is a y strictly below x and strictly above $\mathbf{0}$ such that $(0, y)$ and (y, x) are both empty.

Definition

We call such an x a 2-minimal degree, i.e., a minimal cover of a minimal degree. Similarly an $n + 1$-minimal degree is a minimal cover of an n-minimal degree.

Remark

If we only require that $(0, y)$ to be empty, then the formula corresponds to that x has the minimal bounding property, i.e., x bounds a minimal degree. If we only require that (y, x) to be empty, then the formula corresponds to x being a minimal cover.

Facts

Minimal degrees are "unpowerful":

- not r.e.
- not PA
- not ANR
- $\mathbf{GL}_2$
- not 1-generic
- not 1-random

One might ask whether n-minimal degrees are also unpowerful in some similar way.

Theorem [C.]

There is a 2-minimal degree which is ANR.

Question

Is there a 2-minimal or n-minimal degree which is PA?

Jump classes: (naturally definable in $\mathcal{L}_1 = \{\leq, ', \mathbf{0}\}$)

Definition

A degree $\mathbf{d}$ is low$_n$ $(\mathbf{L}_n)$ if $\mathbf{d}^{(n)} = \mathbf{0}^{(n)}$; $\mathbf{d}$ is high$_n$ $(\mathbf{H}_n)$ if $\mathbf{d}^{(n)} = \mathbf{0}^{(n+1)}$.

Definition

A degree $\mathbf{d}$ is generalized low$_n$ $(\mathbf{GL}_n)$ if $\mathbf{d}^{(n)} = (\mathbf{d} \vee \mathbf{0}')^{(n-1)}$, and $\mathbf{d}$ is generalized high$_n$ $(\mathbf{GH}_n)$ if $\mathbf{d}^{(n)} = (\mathbf{d} \vee \mathbf{0}')^{(n)}$.

Examples of results:

Theorem [Jockusch]

Every $\mathbf{GH}_1$ degree bounds a minimal degree.

Theorem [Lerman]

There is a $\mathbf{GH}_2$ degree which does not bound minimal degrees.

Proposition [Downey, Jockusch, Stob]

Every array recursive (i.e., not ANR) degree is $\mathbf{GL}_2$.

Theorem [Jockusch, Posner]

Every minimal degree is $\mathbf{GL}_2$ $(\mathbf{d}'' = (\mathbf{d} \vee \mathbf{0}')')$.

Proposition/Question [Lerman]

Every n-minimal degree below $\mathbf{0}'$ is $\mathbf{GL}_2$, hence $\mathbf{L}_2$. Is this true for n-minimal degrees in general?

Plan: build a 2-minimal degree which is $\overline{\mathbf{GL}_2}$.

Proposition

Suppose $\mathbf{a}$ is $\mathbf{GL_2}$ and $\mathbf{b}$ is a minimal cover of $\mathbf{a}$, then $\mathbf{b}$ is also $\mathbf{GL_2}$ if either of the following holds:

1. $\mathbf{b} < \mathbf{a}'$, or
2. $\mathbf{b}$ is hyperimmune-free relative to $\mathbf{a}$, i.e., every function recursive in $\mathbf{b}$ is dominated by a function recursive in $\mathbf{a}$.

This means that, in order to build an n-minimal degree which is not $\mathbf{GL}_2$, one has to use a relativized construction of "a hyperimmune minimal degree not recursive in $\mathbf{0}'$".

Our direct construction of a hyperimmune minimal degree

Basic tree component, block, with a guessing feature:

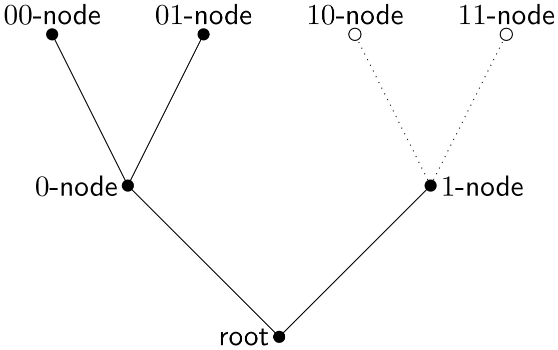

Figure: A guessing block

The 1-node has two successors if and only if some $\varphi_e(e)$ converges.

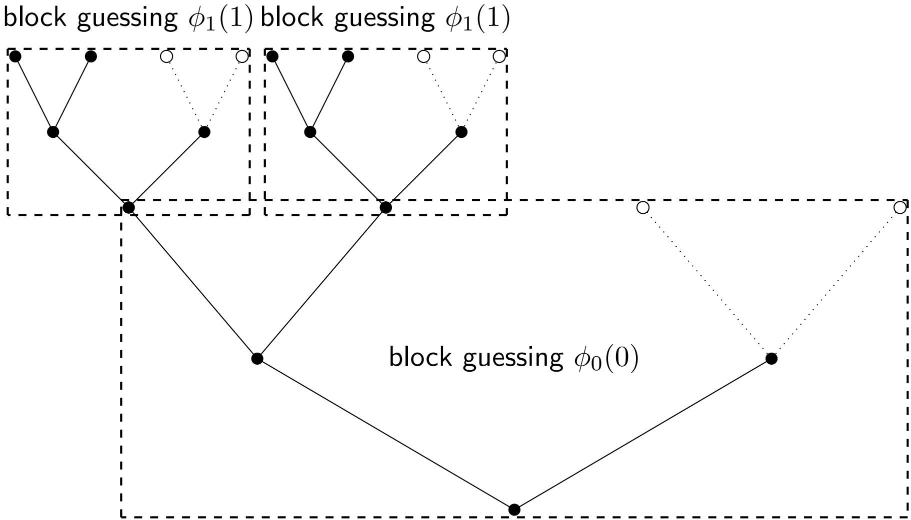

Figure: block structure on a tree

Proposition

If $\mathbf{a}$ is $\mathbf{GL}_1$ and $\mathbf{b}$ is a minimal cover of $\mathbf{a}$, then $\mathbf{b}$ is $\mathbf{GL}_2$.

Proof.

$\mathbf{b}'' = (\mathbf{b} \vee \mathbf{a}')' = (\mathbf{b} \vee \mathbf{a} \vee \mathbf{0}')' = (\mathbf{b} \vee \mathbf{0}')'$. □

This means that we need to make $\mathbf{a}$ $\overline{\mathbf{GL}_1}$ if we want to make $\mathbf{b}$ $\overline{\mathbf{GL}_2}$.

Theorem [Sasso]

There is a minimal degree in $\overline{\mathbf{GL}_1}$.

Sasso used a notion of narrow subtrees to diagonalize against "$\mathbf{GL}_1$". "Path A is on narrow subtree of T" is $\Pi_1(A)$, and we can do some diagonalization:

$$\varphi_e^{A \oplus 0'} \neq A'.$$

$\overline{\mathbf{GL}_2}$

Our plan: build A which is minimal and B which is a minimal cover of A such that for any e there is an x that

$$\varphi_e^{(B\oplus 0')'}(x) \neq B''(x).$$

$$\lim_s \varphi_e^{B\oplus 0'}(x,s) \neq (B \oplus A')'(x).$$

Left hand side:

force the limit to change until we cannot change it.

Right hand side:

find "narrow subtrees" which correspond to $(B \oplus A')'$ questions.

$\mathbf{GH}_2$

In order to make $B'' \equiv_T (B \oplus 0')''$, we want to decide $Tot(B \oplus 0')$, the totality problem for oracle $B \oplus 0'$, using B''.

Idea

In the construction, we try to force $\varphi_e^{B \oplus 0'}$ to be total step by step until we cannot continue (at which step we forced nontotality). Using a similar idea as in $\overline{\mathbf{GL}_2}$ construction, one can code this information of totality into some tree structure which we can retrieve using B''.

$\mathbf{GH}_1$

We want to make $B' \equiv_T (B \oplus 0')'$, i.e., using B' to decide the halting problem with oracle $B \oplus 0'$.

Idea

Use partial trees instead of total trees and make B' compute the whole construction (therefore decide whether we have forced the jump of $B \oplus 0'$.)

Note that this is the highest jump class we can reach by finite iterations of minimality.

Possible Future Work

- relations between n-minimality and other properties such as PA or random
- cones of "relative 2-minimality (with some extra restraint)"

Thank you!