Computable Analysis in the Weihrauch Lattice

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Outline

Equivalence of Theorems

In many mathematical texts one can find statements like the following:

- "In fact, the closed graph theorem, the open mapping theorem and the bounded inverse theorem are all equivalent".

  (Wikipedia, Closed graph theorem, 23 June 2011)

- "Lemma 8.36. The open mapping theorem, the bounded inverse theorem, and the closed graph theorem are equivalent."

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There is an obvious evolution of objects that are considered in mathematical spaces:

- Numbers (set theory)
- Functions (functional analysis)
- Theorems (metamathematics)

Hence, in *metamathematics* as understood here, theorems should be *points* in a space that is subject to usual mathematical investigations, using topology, computability theory etc.

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Computable Metamathematics

- We describe results in a new programme of computable metamathematics.
- Theorems are considered as *points* in a suitable space.
- The location of a theorem in this space reveals insights into the computational content of this theorem.
- The space itself can be studied using techniques of computability theory, topology, descriptive set theory, algorithmic randomness, etc.
- The results are mostly *compatible* with reverse mathematics, but *more informative* as far as the computational content of theorems is concerned.
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Theorems as Multi-Valued Functions

Theorem (Bolzano-Weierstraß Theorem)

Every sequence $(x_n)_{n\in\mathbb{N}}$ in a compact subset $K \subseteq \mathbb{R}$ has a cluster point $x \in \mathbb{R}$.

- This theorem can be represented by the multi-valued map

$$\mathrm{BWT} :\subseteq \mathbb{R}^{\mathbb{N}} \rightrightarrows \mathbb{R}, (x_n) \mapsto \{x \in \mathbb{R} : x \text{ cluster point of } (x_n)\}$$

  with $\mathrm{dom}(\mathrm{BWT}) := \{(x_n) : \overline{\{x_n : n \in \mathbb{N}\}} \text{ compact}\}$.
- By BWT_X we denote the Bolzano-Weierstraß Theorem of space X, defined analogously.
- By CL_X we denote the *cluster point problem* of X (same definition as BWT, but no restriction on the domain).
- Similarly, Weak Kőnig's Lemma can be represented as a map $\mathrm{WKL} :\subseteq \mathrm{Tr} \rightrightarrows \{0,1\}^{\mathbb{N}}$, where Tr denotes the set of binary trees $T \subseteq \{0,1\}^*$ and $\mathrm{dom}(\mathrm{WKL})$ consists of all infinite binary trees.

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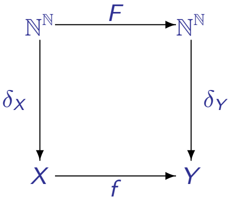
Realizer

Definition

A multi-valued function $f :\subseteq X \rightrightarrows Y$ on represented spaces (X, δ_X) and (Y, δ_Y) is realized by a function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ if

$$\delta_Y F(p) \in f \delta_X(p)$$

for all $p \in \mathrm{dom}(f\delta_X)$. We write $F \vdash f$ in this situation.



Weihrauch Reducibility

Definition (Weihrauch 1990)

Let f and g be multi-valued maps on represented spaces.

- $f \leq_{\mathrm{sW}} g$ (f strongly Weihrauch reducible to g), if there are computable functions $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for all G

$$G \vdash g \implies HGK \vdash f.$$

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That means that there is a uniform way to transform each realizer G of g into a realizer F of f in the given way.

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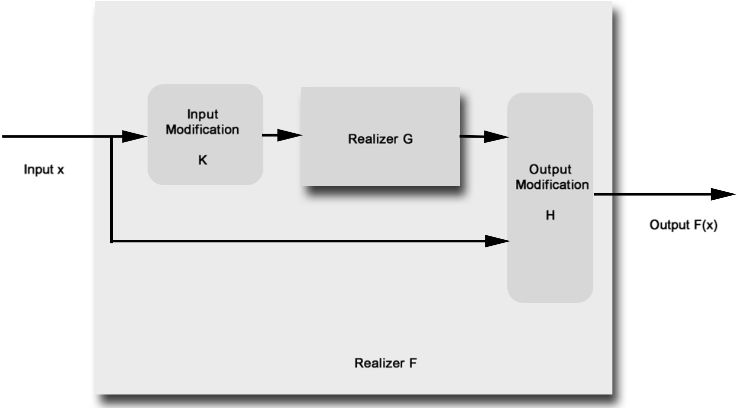

- $F(x) = H\langle x, GK(x)\rangle$ for all admissible inputs x.

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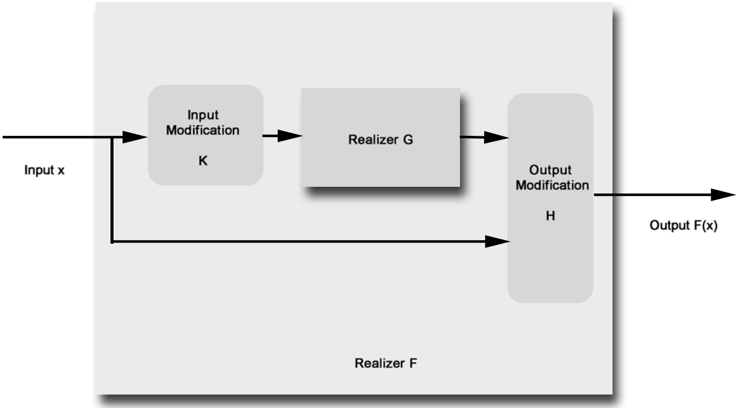

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Algebraic Operations in the Weihrauch Lattice

Definition

Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq W \rightrightarrows Z$ be multi-valued maps. Then we consider the natural operations

- $f \times g :\subseteq X \times W \rightrightarrows Y \times Z$ (product)
- $f \sqcup g :\subseteq X \sqcup W \rightrightarrows Y \sqcup Z$ (coproduct)
- $f \sqcap g :\subseteq X \times W \rightrightarrows Y \sqcup Z$ (sum)
- $f^* :\subseteq X^* \rightrightarrows Y^*, f^* = \bigsqcup_{i=0}^{\infty} f^i$ (star)
- $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}, \widehat{f} = \times_{i=0}^{\infty} f$ (parallelization)

Theorem (B. and Gherardi, Pauly 2009)

Weihrauch reducibility induces a (bounded) lattice with the sum $\sqcap$ as infimum and the coproduct $\sqcup$ as supremum and parallelization and the star operation as closure operators.

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The Choice Operation

Definition

For every represented space X we define the choice operation

$$\mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$$

Here $\mathcal{A}_-(X) := \{A \subseteq X : A \text{ closed}\}$ is the hyperspace of closed subsets with respect to negative information (the upper Fell topology = dual of the Scott topology). We write K_X if $\mathcal{A}_-$ is replaced by $\mathcal{K}_-$ (compact subsets).

That is, choice C_X is an operation that takes as input a description of what does *not* constitute a solution and has to find a solution.

Lemma

- $C_\emptyset \equiv_{\mathrm{W}} \mathsf{K}_\emptyset \equiv_{\mathrm{W}} \mathbf{0}$.
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Binary Choice and LLPO

Example

- Binary choice $\mathbf{2} = \mathsf{C}_{\{0,1\}}$ could receive as a potential input:

$$\bot, \bot, \bot, 1, 1, \bot, 1, 1, 1, \ldots$$

- Here $\bot$ stands for "no information". As soon as the information 1 appears, it is clear that the only possible remaining choice is 0.
- This is similar to the "lesser limited principle of omniscience" LLPO.

Proposition

$$\mathsf{LLPO} \equiv_{\mathrm{W}} \mathbf{2} \equiv_{\mathrm{W}} \mathsf{K}_{\{0,1\}} \quad \textit{and} \quad \mathsf{LLPO}^* \equiv_{\mathrm{W}} \mathbf{2}^* \equiv_{\mathrm{W}} \mathsf{K}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}.$$

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Choice on Cantor Space

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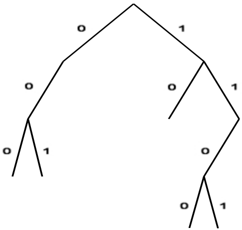
- Cantor choice $\mathsf{C}_{\{0,1\}^{\mathbb{N}}}$ could receive as a potential input a sequence of finite words:

$$0111000, 01000, 010100001111000, \ldots$$

- The goal is to find an infinite word that does not have any of these words as prefix.



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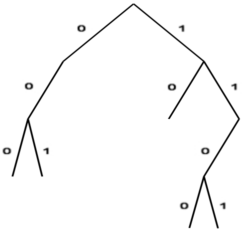

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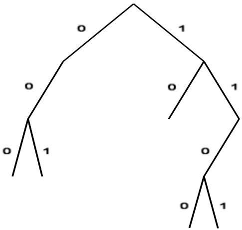

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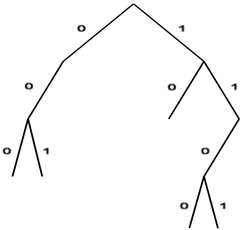

Choice on Cantor Space

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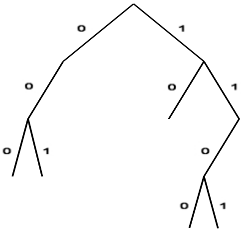

Weak Kőnig's Lemma and Cantor Choice

Theorem

$$\mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{C}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{K}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{W}} \widehat{\mathsf{C}_{\{0,1\}}} = \widehat{\mathbf{2}}.$$

Theorem (B. and Gherardi 2009)

The following are Weihrauch equivalent:

1. *Cantor Choice* $\mathsf{C}_{\{0,1\}^{\mathbb{N}}}$.
2. *Compact Choice* C_X *for each computably compact computable metric space* X *without isolated points.*
3. *Weak Kőnig's Lemma.*
4. *The Hahn-Banach Theorem (Gherardi, Marcone 2009).*

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Natural Choice and Finitely Many Mind Changes

Example

- Natural number choice $C_{\mathbb{N}}$ could receive as a potential input:

$$5, 112, 3, 5, 23, 0, 42, 1, 25, \ldots$$

- This is a discontinuous operation, however, it can be computed with finitely many mind changes.

Theorem (B., de Brecht and Pauly 2010)

For all f the following statements are equivalent:

- $f \leq_W C_{\mathbb{N}}$
- *f is computable with finitely many mind changes.*

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The Baire Category Theorem and Discrete Choice

Theorem (B. and Gherardi 2009)

The following are Weihrauch equivalent:

1. *Discrete Choice* $\mathsf{C}_{\mathbb{N}}$.
2. *The Baire Category Theorem (for each complete computable metric space* X *and each sequence* $(A_i)_{i\in\mathbb{N}}$ *of closed subsets with* $X=\bigcup_{i=0}^{\infty} A_i$ *there exists an* $n\in\mathbb{N}$ *such that* $A_n^{\circ}\neq\emptyset$*).*
3. *Banach's Inverse Mapping Theorem.*
4. *Closed Graph Theorem.*
5. *Open Mapping Theorem.*

Definition

Let X be a non-empty computable metric space. We define

$$\mathsf{BCT} :\subseteq \mathcal{A}_-(X)^{\mathbb{N}} \rightrightarrows \mathbb{N}, (A_i)_{i\in\mathbb{N}} \mapsto \{n\in\mathbb{N} : A_n^{\circ}\neq\emptyset\}$$

with $\mathrm{dom}(\mathsf{BCT}) = \{(A_i)_{i\in\mathbb{N}} : X=\bigcup_{i=0}^{\infty} A_i\}$.

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The Baire Category Theorem and Discrete Choice

Proof.

Proof idea for $\mathsf{BCT} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$.
"$\mathsf{BCT} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$" Given (A_i), the set

$$\{\langle k, n\rangle : \emptyset \neq B_k \subseteq A_n\}$$

is co-c.e. in all parameters. Hence one can find a number $\langle k, n\rangle$ in this set using $\mathsf{C}_{\mathbb{N}}$. In this case $n \in \mathsf{BCT}(A_i)$.
"$\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{BCT}$" Given a sequence $(n_i)_{i\in\mathbb{N}}$ that enumerates a set of natural numbers, we compute the sequence (A_i) of closed subsets $A_i \subseteq X$ with

$$A_i := \begin{cases} \emptyset & \text{if } (\exists i)\ n = n_i \\ X & \text{otherwise} \end{cases}$$

This sequence is computable in (n_i) and each $n \in \mathsf{BCT}(n_i)$ has the property that n does not appear in (n_i). $\square$

Proof.

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Intermediate Value Theorem and Connected Choice

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The following are Weihrauch equivalent:

1. *Connected Choice* $\mathsf{CC}_{[0,1]}$.
2. *The Intermediate Value Theorem.*

Theorem (B. and Gherardi 2009)

Connected Choice $\mathsf{CC}_{[0,1]}$ *and Discrete Choice* $\mathsf{C}_{\mathbb{N}}$ *are incomparable in the Weihrauch lattice.*

Proof.

$\mathsf{C}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{CC}[0,1]$ follows with lattice theoretic arguments.
$\mathsf{CC}_{[0,1]} \not\leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$ can be proved with the help of the Baire Category Theorem. □

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Reverse Mathematics

In Reverse Mathematics all the following theorems are provable over RCA_0:

- The Intermediate Value Theorem.
- The Baire Category Theorem.
- The Open Mapping Theorem.
- The Closed Graph Theorem.
- Banach's Inverse Mapping Theorem.

Parallelization of Discrete Choice

Theorem

The following are Weihrauch equivalent:

1. *Parallelization of discrete Choice* $\widehat{\mathsf{C}_{\mathbb{N}}}$.
2. *The limit operation* $\lim$ *on* $\mathbb{R}$ *or* $\mathbb{N}^{\mathbb{N}}$.
3. *The Monotone Convergence Theorem (B., Gherardi and Marcone 2011).*
4. *The Fréchet-Riesz Theorem for Hilbert Spaces (B. and Yoshikawa 2008).*
5. *The Radon-Nikodym Theorem (Hoyrup, Rojas and Weihrauch 2011).*

Choice and Classes of Computability

Theorem (B., de Brecht and Pauly 2010)

The following operations are complete in the Weihrauch lattice for the respective classes of functions:

Choice	**Class of functions**
$\mathsf{C}_{\{0\}}$	*computable*
$\mathsf{C}_{\mathbb{N}}$	*computable with finitely many mind changes*
$\mathsf{C}_{\{0,1\}^{\mathbb{N}}}$	*weakly computable*
$\widehat{\mathsf{C}_{\mathbb{N}}}$	*limit computable (effectively Σ^0_2–measurable)*
$\widehat{\mathsf{C}_{\mathbb{N}}}^{\circ k}$	*effectively Σ^0_{k+1}–measurable*
$\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$	*effectively Borel measurable*
C_A	*non-deterministically computable with advice space $A \subseteq \mathbb{N}^{\mathbb{N}}$*

The Uniform Low Basis Theorem

Theorem (B., de Brecht and Pauly 2010)

$C_{\mathbb{R}}$ *is low computable.*

Corollary (Low Basis Theorem of Jockusch and Soare)

Each co-c.e. closed subset $A \subseteq \{0,1\}^{\mathbb{N}}$ *has a low point* $p \in A$, *i.e. a point such that* $p' \leq_{\mathrm{T}} \emptyset'$.

Theorem

For all f *the following statements are equivalent:*

- $f \leq_{\mathrm{sW}} \mathsf{L} = J^{-1} \circ \lim$
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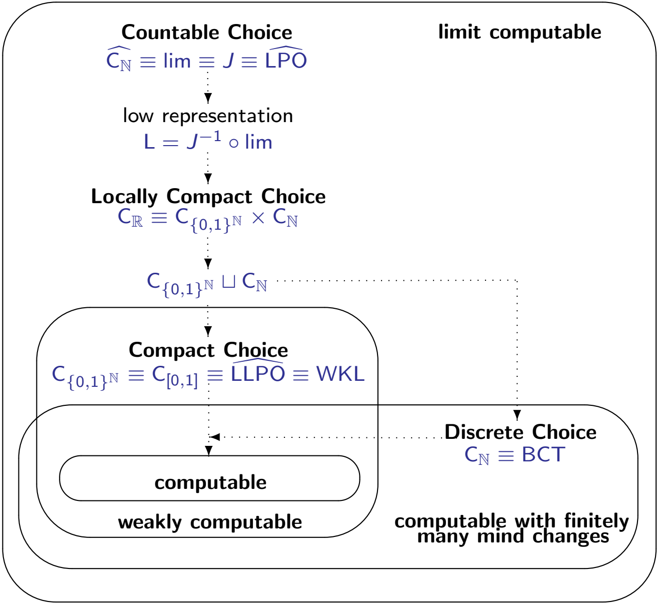
Choice in the Weihrauch Lattice



The Jump/Derivative in the Weihrauch Lattice

Definition

Let $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ be a multi-valued function on represented spaces. Then the *derivative* or *jump* f' of f is the function $f' :\subseteq (X, \delta'_X) \rightrightarrows (Y, \delta_Y)$. Here $\delta' := \delta \circ \lim$.

Example

We obtain the following:

1. $\mathsf{C}'_\emptyset \equiv_{\mathrm{sW}} \mathsf{C}_\emptyset$,
2. $\mathsf{C}'_{\{0\}} \equiv_{\mathrm{sW}} \mathsf{C}_{\{0\}}$,
3. $\mathrm{id}'_X \equiv_{\mathrm{sW}} \lim_X$,
4. $\lim' \equiv_{\mathrm{sW}} \lim \circ \lim$,
5. $(J^{-1})' \equiv_{\mathrm{sW}} J^{-1} \circ \lim = \mathsf{L}$,
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Properties of the Derivative

Proposition

Let f, g be multi-valued functions on represented spaces. Then:

1. $f \leq_{\mathrm{sW}} f'$, $f \leq_{\mathrm{sW}} g \Longrightarrow f' \leq_{\mathrm{sW}} g'$,
2. $f \circ g' = (f \circ g)'$, $f' \times g' \equiv_{\mathrm{sW}} (f \times g)'$,
3. $\widehat{f'} \equiv_{\mathrm{sW}} (\widehat{f}\,)'$, $f'^{*} \leq_{\mathrm{sW}} f^{*\prime}$,
4. $f' \sqcap g' \equiv_{\mathrm{sW}} (f \sqcap g)'$, $f' \sqcup g' \leq_{\mathrm{sW}} (f \sqcup g)'$.

Theorem (B., Gherardi and Marcone 2011)

Let f and g be multi-valued functions on represented spaces. If g is a cylinder, then the following are equivalent:

1. $f \leq_{\mathrm{W}} g'$,
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The Cluster Point Problem as Derivative of Choice

Definition (Cluster Point Problem)

Let X be a represented space. We define

$$\mathsf{L}_X : X^{\mathbb{N}} \to \mathcal{A}_-(X), (x_n) \mapsto \{x \in X : x \text{ is cluster point of } (x_n)\}.$$

We call $\mathsf{CL}_X := \mathsf{C}_X \circ \mathsf{L}_X :\subseteq X^{\mathbb{N}} \rightrightarrows X$ the *cluster point problem*.

Proposition

$\mathsf{L}_X \leq_{\mathrm{sW}} \lim$ *for computable metric spaces.*

Proof.

The claim follows from

$$x \notin \mathsf{L}_X(x_n) \iff (\exists i)(x \in B_i \text{ and } (\exists k)(\forall n \geq k)\ x_n \notin B_i). \qquad \square$$

Corollary

$\mathsf{CL}_X \leq_{\mathrm{sW}} \mathsf{C}_X'$.

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The Cluster Point Problem as Derivative of Choice

Theorem (B., Gherardi and Marcone 2011)

$\mathsf{C}'_X \equiv_{\mathrm{sW}} \mathsf{CL}_X$ *for each computable metric space* X*.*

Proof.

It remains to show $\mathsf{C}'_X \leq_{\mathrm{sW}} \mathsf{CL}_X$. That is given a sequence of names of closed sets A_n the limit of which describes A, one needs to compute a cluster point of A.
The idea is to approximate points in A by points that tend to "escape" from the negative descriptions of the sets A_n. □

Corollary (Le Roux and Ziegler for Euclidean space 2008)

Let X *be a computable metric space. Then a set* $A \subseteq X$ *is co-c.e. closed in the limit, if and only if it is the set of cluster points of some computable sequence* (x_n) *in (the dense subset of)* X*.*

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with $\mathrm{dom}(\mathsf{BWT}_X) := \{(x_n) \in X^{\mathbb{N}} : \overline{\{x_n : n \in \mathbb{N}\}} \text{ is compact}\}$.

Theorem (B., Gherardi and Marcone 2011)

$\mathsf{BWT}_X \equiv_{\mathrm{sW}} \mathsf{K}'_X$ *for all computable metric spaces* X.

Corollary

$\mathsf{WKL}' \equiv_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{R}}$.

It is known that instancewise the Bolzano-Weierstraß Theorem is equilvanet to $\boldsymbol{\Sigma}^0_1 - \mathsf{WKL}$ over RCA_0 (Kohlenbach and Safarik 2010).

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$\mathrm{WKL}' \equiv_{\mathrm{sW}} \mathrm{BWT}_{\mathbb{R}}$.

It is known that instancewise the Bolzano-Weierstraß Theorem is equilvanet to $\mathbf{\Sigma}^0_1 - \mathrm{WKL}$ over RCA_0 (Kohlenbach and Safarik 2010).

The Bolzano-Weierstraß Theorem

Definition (Bolzano-Weierstraß Theorem)

Let X be a represented space. Then $\mathsf{BWT}_X :\subseteq X^{\mathbb{N}} \rightrightarrows X$ is defined by

$$\mathsf{BWT}_X(x_n) := \{x \in X : x \text{ is cluster point of } (x_n)\}$$

with $\mathrm{dom}(\mathsf{BWT}_X) := \{(x_n) \in X^{\mathbb{N}} : \overline{\{x_n : n \in \mathbb{N}\}} \text{ is compact}\}$.

Theorem (B., Gherardi and Marcone 2011)

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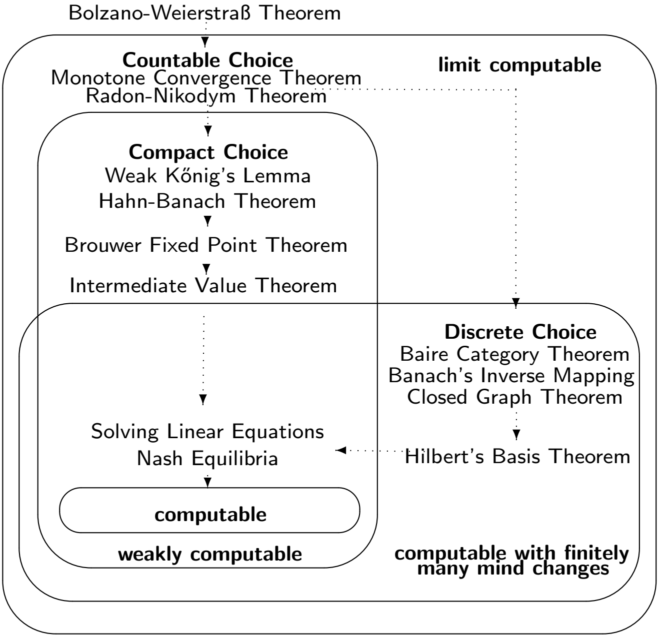
Reverse Computable Analysis



Open Problems

- Is the Weihrauch lattice a Brouwerian algebra (Heyting algebra) in some sense?
- The answer is "no", strictly speaking (Higuchi and Pauly 2011).
- The answer is "yes" for total Weihrauch reducibility (a variant where only total realizers are considered - unpublished work with Guido Gherardi).
- In which sense is the Weihrauch lattice model for some form of (intermediate) logic?
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- Are there interesting theorems on specific higher levels of the effective Borel hierarchy?

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