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From causality to computability: a quantum extension of Gandy's theorem

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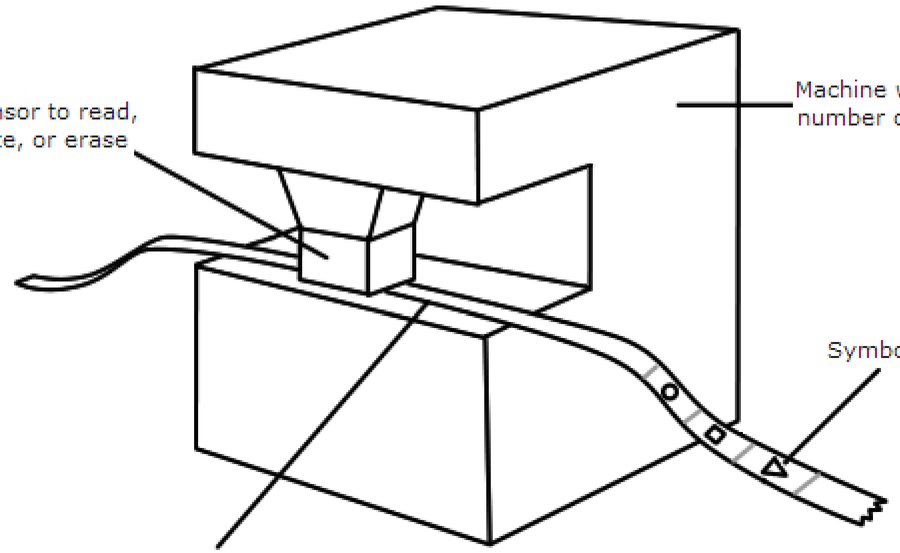
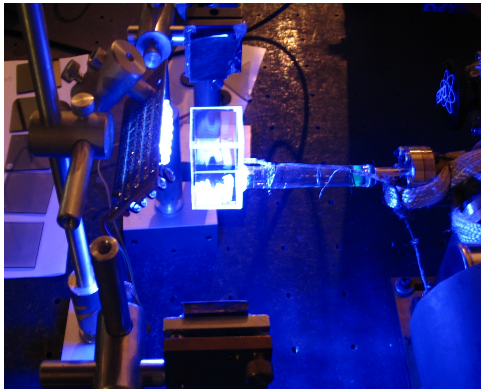


A Turing Machine

(Strong) (physical) Church-Turing thesis: Anything that can be computed (by a physical system) can be computed (efficiently) by a Turing machine.

The Halting function *h*. Physics, and even H/W independence.



~~Strong~~ physical CT thesis.

QC: Theoretical physics helps / shakes Theoretical CS.

Computing process is physical. CS about extracting from physics its ability to run computations.

Physical CT thesis?

Introduction > Deutsch *vs* Nielsen

[Deutsch, Proc. Roy. Soc, 1985]

QC by pen and paper...

QC computable
= TM computable

"Physical CT thesis!" in QC

[Nielsen, "Computable Functions, Quantum Measurements, and Quantum Dynamics", PRL, 1997]

"We construct quantum mechanical (…) unitary operators which, if implemented (…) would contradict the Church-Turing thesis"

~~"Physical CT thesis!"~~ in QT

[Deutsch, Proc. Roy. Soc, 1985]

[Nielsen, "Computable Functions, Quantum Measurements, and Quantum Dynamics", PRL, 1997]

QC by pen and paper...

QC computable
= TM computable

"Physical CT thesis!" in QC

QC < QT.

"*Natural processes may take place in infinte-dimensional state spaces*"

Introduction > Dirac *vs* Nielsen

[Dirac, OUP, 1958]

[Nielsen, “Computable Functions, Quantum Measurements, and Quantum Dynamics”, PRL, 1997]

“In practice it may be very awkward (…) but the theory always allows one to imagine that the measurement can be made.”

“Physical CT thesis!” in QT'

“A more satisfactory program is to address the problem of a sharp characterization of the class of (…) unitary dynamics which may be realized in physical systems. ”

QT = postulates defining states & dynamics upon vec. spaces.

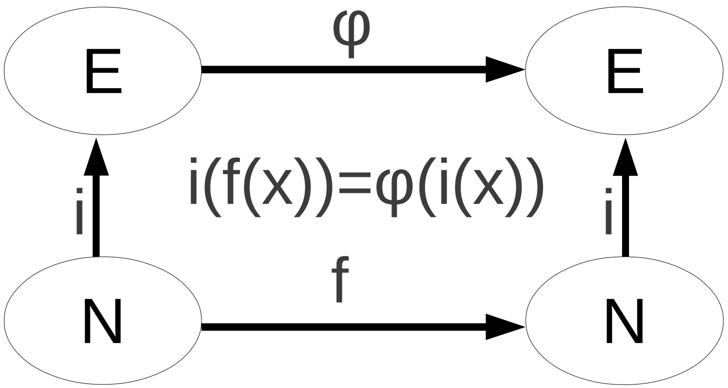
We will learn:

- (To define computability upon vec. spaces. [CIE 2010])
- How physics-like postulates can yield computability [Gandy, Kleene symposium, 1981]

We will move towards:

QT' = postulates that define computable states & dynamics upon vec. Spaces, hence CT thesis compatible.

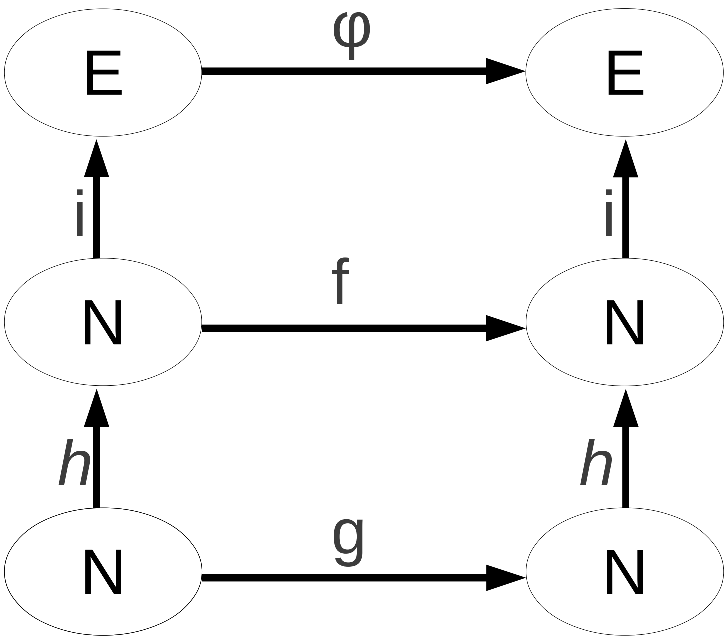
If E countable, define computable functions over E via N:



Indexing i. Partial function from N to E such that

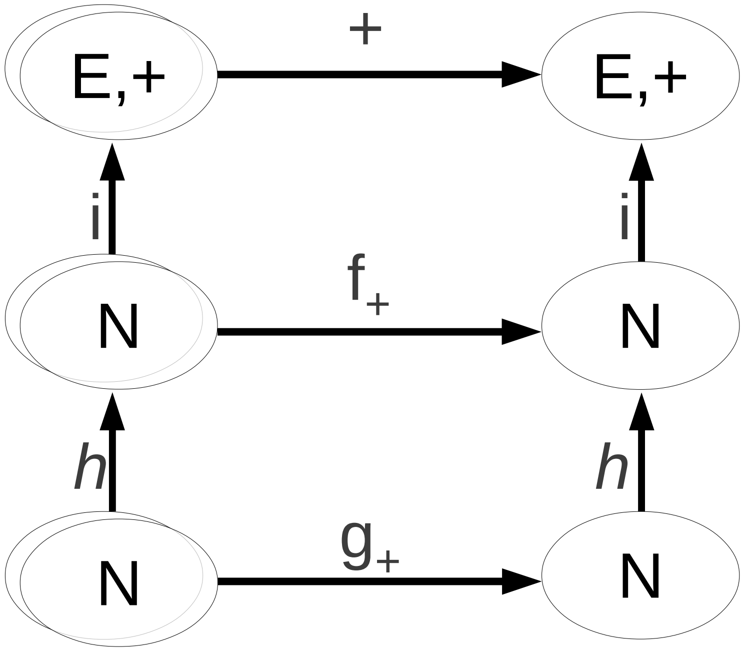
- dom(i) effectively enumerable
- i is surjective
- equality is decidable

But this notion of computability depends on the indexing:



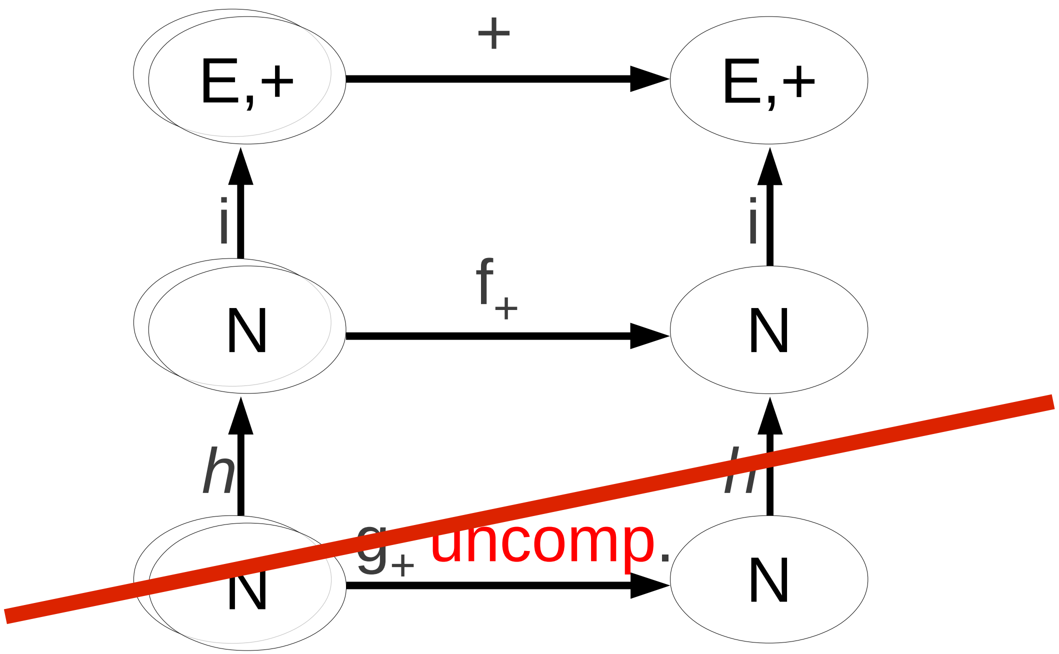
[Montague, Synthese, 1960]

Restricting to admissible indexings...



...

Restricting to admissible indexings...



…can make computability stable. [Rabin, Transac. AMS, 1960]

Computability in vector spaces > Facts

K: a field with stable computablity.

<K, E, +, x, .>: a K-vector space has stable computability...
iff dim(E) is finite.

E: a finite dim. vec. space with o.n.b. $\{|q>,|e_1>,\ldots\}$.

H: an infinite dim. vec. space. with o.n.b. $\{|\ldots qeeeq\ldots>\}$

⊗: a tensor product $|e_i>,|\ldots qeeeq\ldots> \otimes\ |\ldots qe_ieeeq\ldots>$

<K, E, H, +, x, ., ⊗>: has stable computability.

Finite extensions of <Q,+,x> have stable computability.

[CIE 2010]

Homogeneity of space. $\Sigma(\sigma(A))=\sigma(A)$. $G(t,t+T)$ commutes with σ.

Homogeneity of time. $G(t,t+T)$ is independent of t.

Bounded density of information. A finite implies $\Sigma(A)$ finite.

Bounded speed of information. $\rho(A,t+T)=f(\rho(A',t))$.
(with T sufficiently small and A' the area of radius 1 around A.)

Quiescence. All but a finite region is quiescent.

With A: a region of space

$\Sigma(A)$: the state space of A

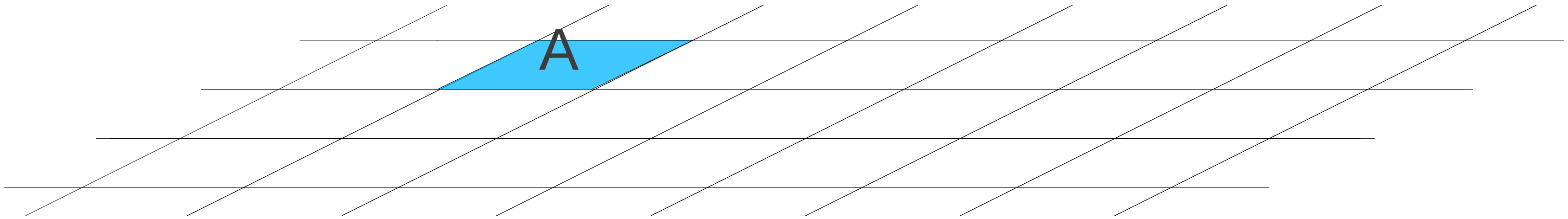
$\rho(A,t)$: the state of A at time t

$G(t, t+T)$: the global evolution from t to t+T

Partition space.

Give: cells of finite size with an indexing, and a set of translations.

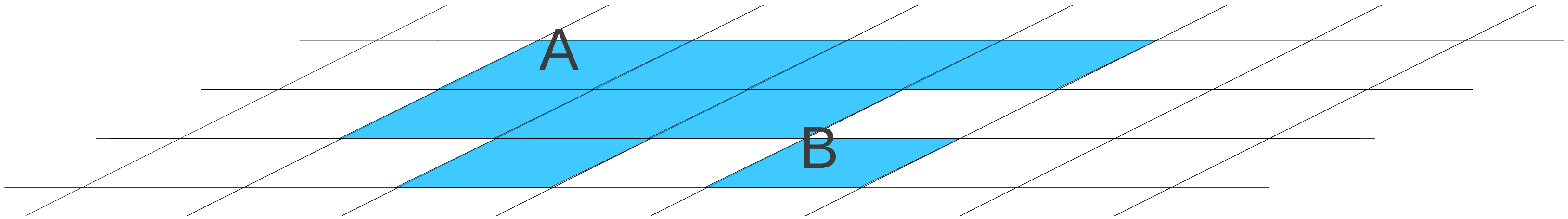
Such that: translations map cells into cells; cells are always related by a translation; area of radius 1 around a cell intersect a finite number of cells whose indexes can be computed from that of the cell.



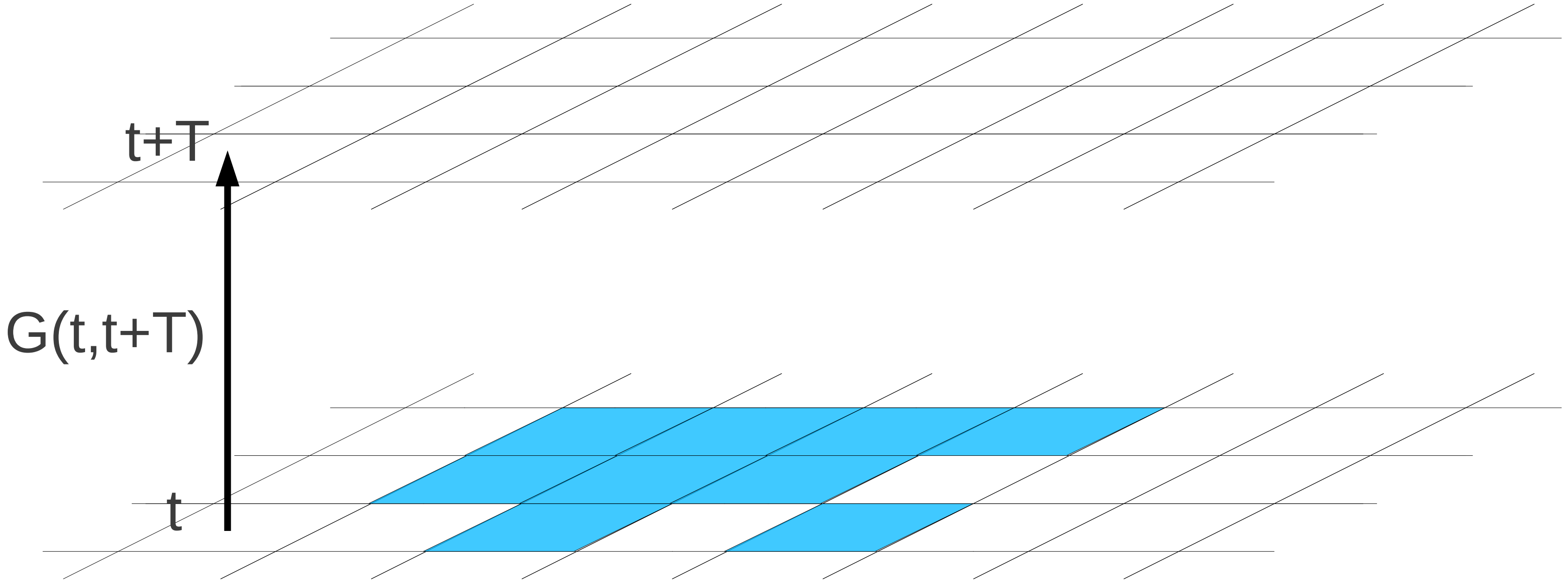
Apply bounded density. Σ(A) finite.

Apply homogeneity of space. Σ(B)=Σ(σ(A))=Σ(A)=S.

Apply quiescence. All but finite number of cells blank.

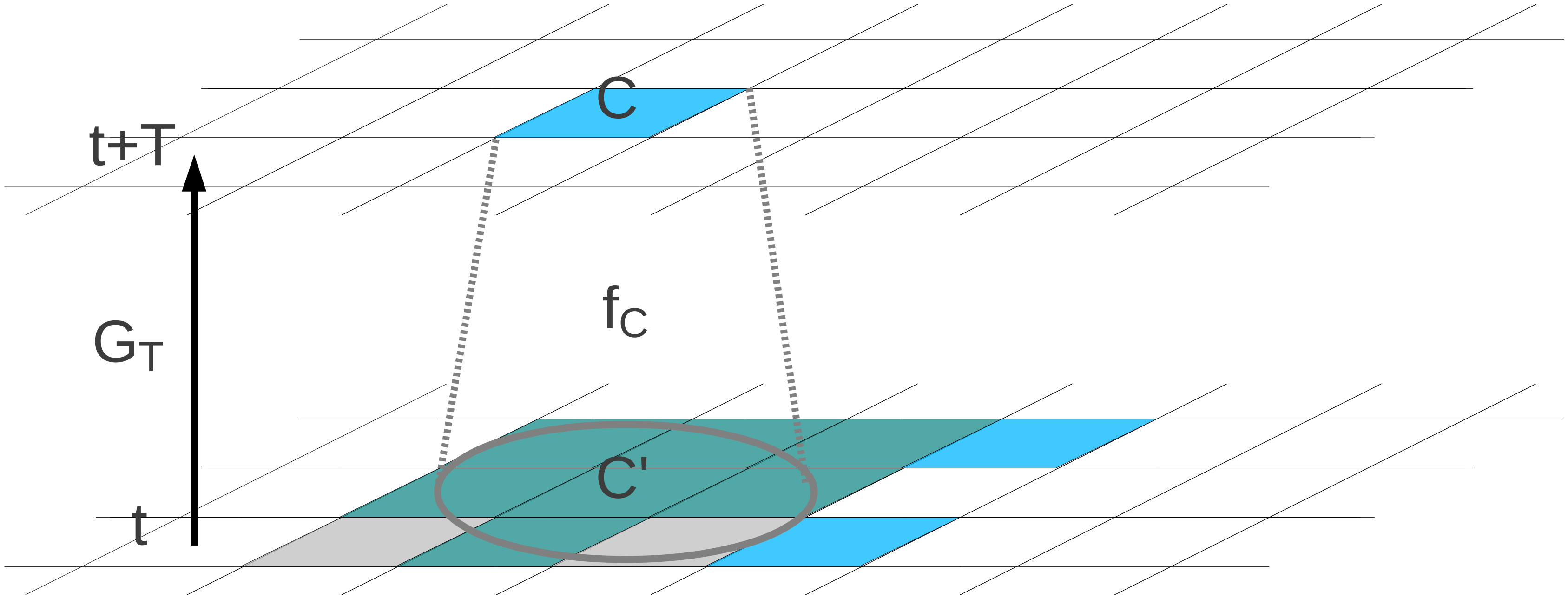


Apply homogeneity of time. G(t,t+T) is a G_T.



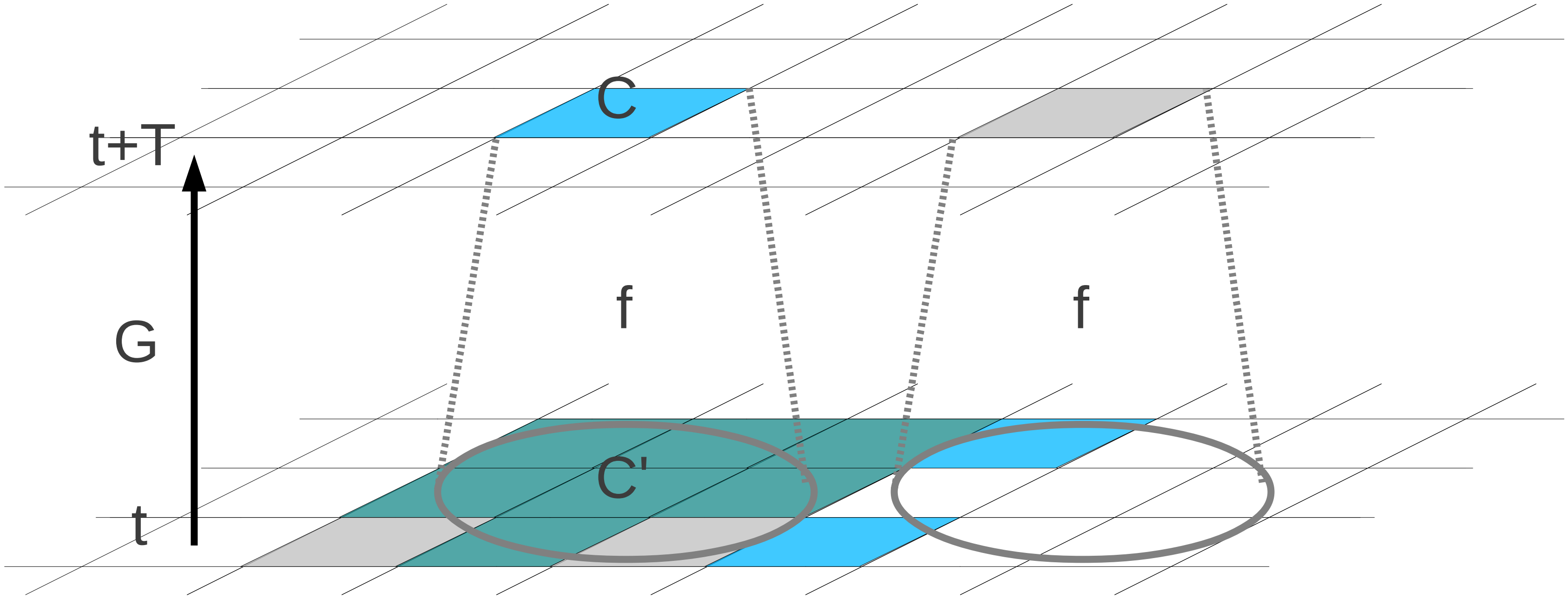
Apply bounded speed of information. Fix a T such that

$$\rho(C,t+T)=f_C(\rho(C',t)).$$



Apply homogeneity of space again. f_C is an f.

f is $S^9 \rightarrow S$ hence computable, hence so is G.



Important idea that causality yields computability.

But some naive (they have been criticized) hypotheses:

- Bounded density of information *vs* superpositions
- Bounded speed of information *vs* entanglement

Homogeneity of space. $\Sigma(\sigma(A))=\sigma(A)$. $G(t,t+T)$ commutes with σ.

Homogeneity of time. $G(t,t+T)$ is independent of t.

Bounded density of information. A finite implies $\Sigma(A)$ a finite dim. vector space over a finite extension of Q.

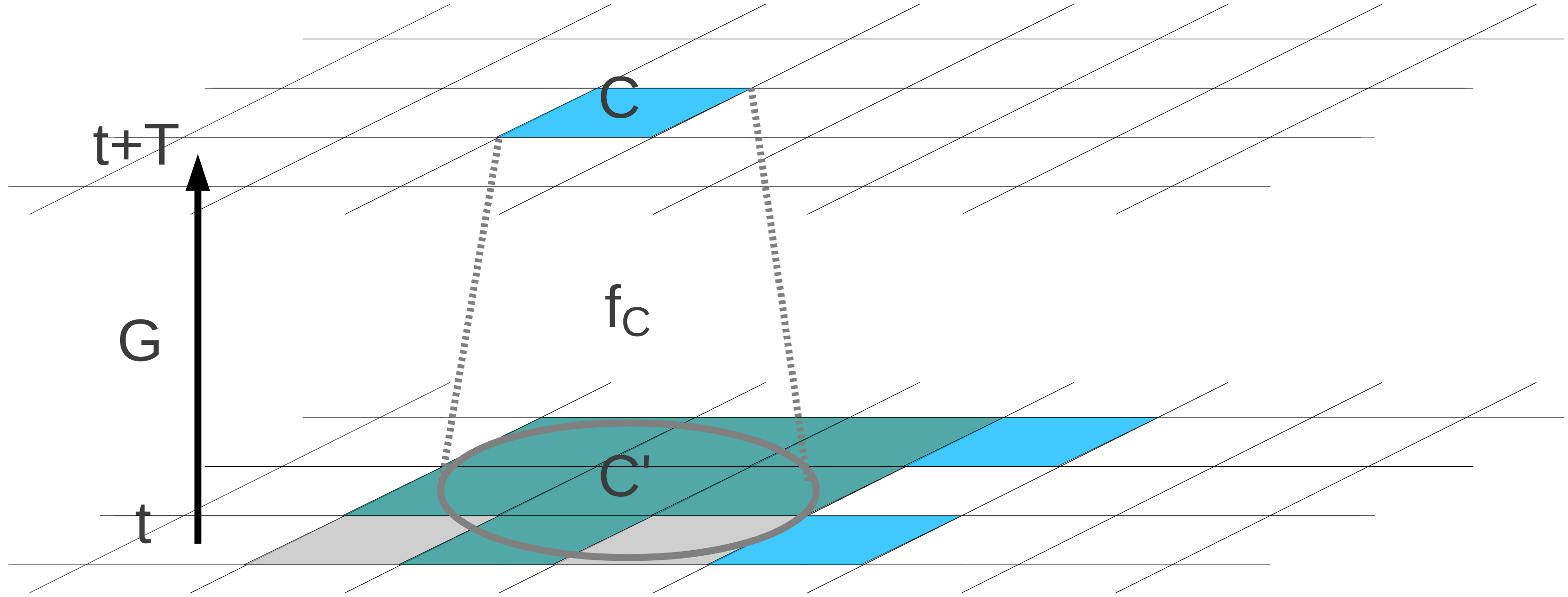
Bounded speed of information. $\rho(A,t+T)=f(\rho(A',t))$.
(with T sufficiently small and A' the area of radius 1 around A.)

Quiescence and unitarity. All but a finite region is quiescent.
$G(t,t+T)$ is unitary.

With $\rho(A,t)$: the reduced density matrix over A at time t.

(...)

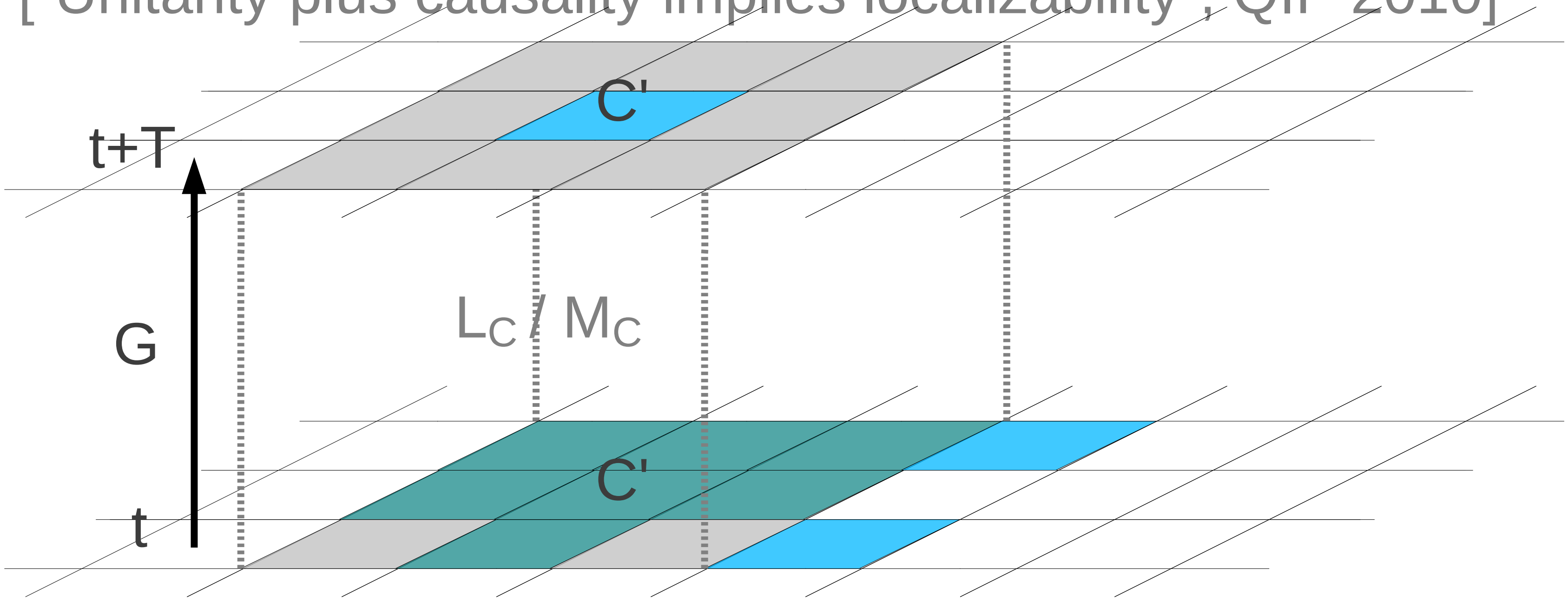
G is unitary, and causal: $\rho(C,t+T)=f_C(\rho(C',t))$.



G is unitary, and causal: $\rho(C,t+T)=f_C(\rho(C',t))$.

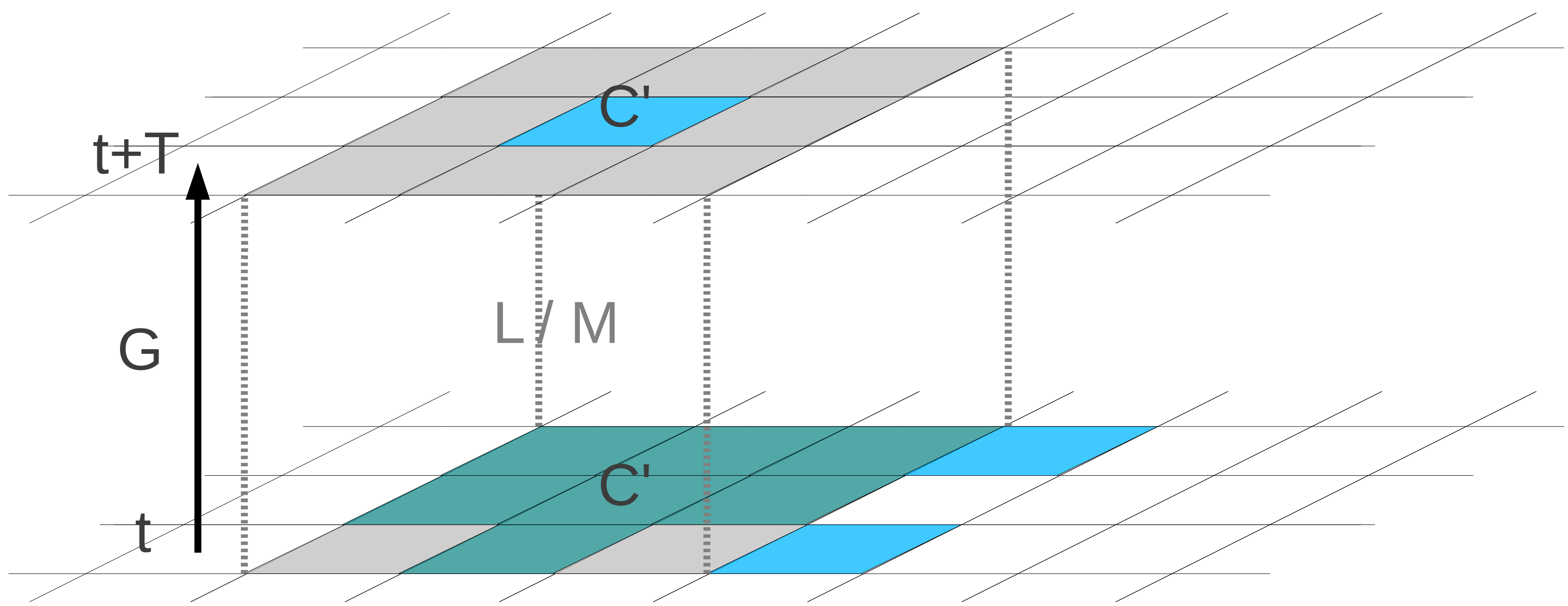
$G = \prod_C M_C \prod_C L_C$, with L_C, M_C linear operators localized upon C'.

cf. [“Unitarity plus causality implies localizability”, QIP 2010]



Apply homogeneity of space. $G = \prod_{C} M \prod_{C} L$.

Apply quiescence. $G = \prod_{C \in \blacksquare} M \prod_{C \in \blacksquare} L$.

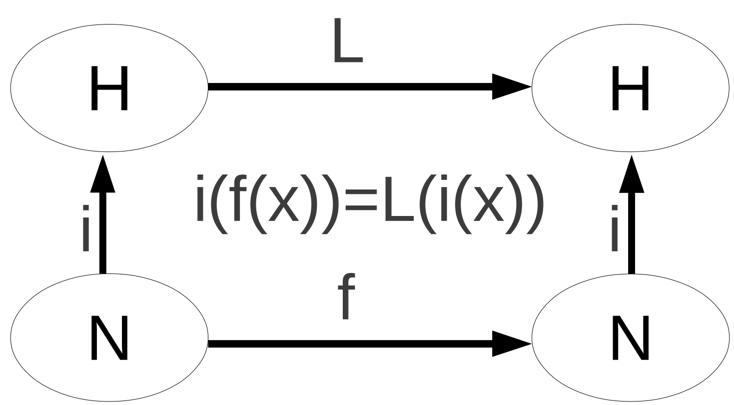


Gandy the quantum > Proof outline

Apply bounded density. Σ(C) is a finite dim. vector space over K a finite extension of Q. Call it E.

<K, E, H, +, x, ., ⊗> has stable computability.

Lemma. Local linear maps are computable.



M,L are local linear maps. Hence the computability of

$$G = \prod_{C\in} M \prod_{C\in} L.$$

Gandy the quantum > Necessity

W/o homogeneity of space. $G_i = \text{NOT}^{h(i)}$.

W/o homogeneity of time. $G(t,t+iT) = \otimes \text{NOT}^{h(i)}$.

W/o bounded density of information. Classical G_i: $x \to 2x+h(i)$.

Quantum G_i: $|0\rangle \to \sqrt{h}|0\rangle+\sqrt{(1-h)}|1\rangle$, $|1\rangle \to \sqrt{(1-h)}|0\rangle-\sqrt{h}|1\rangle$.

W/o bounded speed of information. $G_{i,\ldots}$: $\ldots 01^j 0 \ldots \to h(j)\ldots$

W/o quiescence and unitarity. Classical: the input can be uncomputable. Quantum: w/o unitarity, we have quantum operations and hence stochastic maps. G_i: $x_i \to$ unif. random, but $G_{h(0),h(1),\ldots}$: $x_{h(0)},x_{h(1)},\ldots$ □ correlated.

Conclusion > Physics symmetries and the CT thesis.

Gandy. Physics symmetries can buy you the CT thesis.

Result: a quantum extension of Gandy's theorem.

Nielsen. CT thesis as a symmetry of physics.

Result: the premices of a computable QT.

Why just premices?

- Observables? OK, à la Asher Peres.
- Bounded density? A question for physicists.
- Continuous time, space, isotropy, further symmetries? A question for us.