Phase transitions related to the pigeonhole principle

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Since Paris introduced them in the late seventies, densities turned out to be fruitful for studying independence results, as they often generate strength. We investigate the complexity of two densities which are strongly related to the pigeonhole principle, a well-know combinatorial principle. It is a finite instance of $\forall k\, \mathrm{RT}^1_k$, with RT^n_k denoting Ramsey's theorem for n dimensions and k colours. More precisely, the aim is to miniaturise $\forall k\, \mathrm{RT}^1_k$ and RT^1_2 by defining PHP-density and PHP2-density. In addition, both densities depend on a parameter function $f\colon \mathbb{N} \to \mathbb{N}$. We define two first-order assertions and study their provability with respect to $\mathrm{I}\Sigma_1$, the first-order part of RCA_0.

In the case of PHP-density it turns out that $f(i) = i^{\frac{1}{A_d^{-1}(i)}}$, with d a natural number, gives rise to no more than primitive recursive growth, in contrast to $f(i) = i^{\frac{1}{A_\omega^{-1}(i)}}$, which leads to Ackermannian growth. Here A_d denotes the dth branch of the Ackermann function A_ω. In the case of PHP2-density, Ackermannian growth is obtained for $f(i) = \frac{1}{A_\omega^{-1}(i)} \log(i)$, whereas for $f(i) = \frac{1}{A_d^{-1}(i)} \log(i)$ it is not.

Note that the PHP-density threshold functions are exactly the same as those for the parameterised Kanamori-McAloon principle, whereas the PHP2-density functions equal those for the parameterised Paris-Harrington principle.

Keywords Ackermann function, pigeonhole principle, Ramsey theory, phase transitions, primitive recursive growth.

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