GOODNESS AND JUMP INVERSION IN THE ENUMERATION DEGREES.

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In this talk I will review recent work relating to jump inversion techniques and their application in the enumeration degrees. Underlying this research is, on the one hand the notion of a good approximation and, on the other, a fundamental characterisation of the enumeration jump in terms of index sets.

Definition 1.1 ([LS92, Har10])**.** A uniformly computable enumeration of finite sets $\{X_s\}_{s\in\omega}$ is said to be a *good approximation* to the set X if:

(1) $\forall s\,(\exists t \geq s)[\,X_t \subseteq X\,]$

(2) $\forall x\,[\,x \in X \quad \text{iff} \quad \exists t\,(\forall s \geq t)[\,X_s \subseteq X \Rightarrow x \in X_s\,]\,]$.

In this case we say that X is *good approximable.* An enumeration degree $\boldsymbol{a}$ is said to be *good* if it contains a good approximable set. Otherwise it is said to be *bad.*

Definition 1.2. A set B is said to be *jump uniform under* $\leq_{\mathrm{e}}$ if, for any set A,

$$A \leq_{\mathrm{e}} J_B \quad \Leftrightarrow \quad \exists X[\,X \leq_{\mathrm{e}} B \;\;\&\;\; A = \{\,e \mid X^{[e]} \text{ is finite}\,\}\,] \tag{1.1}$$

where J_B is notation for the enumeration jump of B and $X^{[e]}$ notation for the e^{th} column of X.

Note 1.3. Griffith proved in [Gri03] that ($\Leftarrow$) holds for *any* set B whereas ($\Rightarrow$) holds provided that $deg_{\mathrm{e}}(B)$ is total (i.e. contains a total function). However, it turns out that ($\Rightarrow$) holds in the more general case of $deg_{\mathrm{e}}(B)$ being good [Har10].

The notion of jump uniformity can be used directly to prove that, for any enumeration degrees $\boldsymbol{a} < \boldsymbol{b}$ such that $\boldsymbol{b}$ is good there exists a degree $\boldsymbol{a} \leq \boldsymbol{c} < \boldsymbol{b}$ such that $\boldsymbol{b}' = \boldsymbol{c}'$ [Gri03, Har10]. Jump uniformity techniques are also particularly suitable for the study of the distribution of the local noncuppable enumeration degrees and of the properly Σ^0_2 enumeration degrees. (An enumeration degree $\boldsymbol{a} < \boldsymbol{0}'_{\mathrm{e}}$ is *noncuppable* if, for all $\boldsymbol{y} < \boldsymbol{0}'_{\mathrm{e}}$, $\boldsymbol{a} \cup \boldsymbol{y} \neq \boldsymbol{0}'_{\mathrm{e}}$ and is *properly* Σ^0_2 if it contains no Δ^0_2 set.) Indeed, combined with a construction using the Turing Halting set $\mathcal{K}$ as oracle, Cooper and Copestake's results on the distribution of the properly Σ^0_2 enumeration degrees [CC88] can be extended by showing, using only a finite injury argument, that there exists a high (i.e. $\boldsymbol{a}'_{\mathrm{e}} = \boldsymbol{0}'_{\mathrm{e}}$) enumeration degree $\boldsymbol{a} < \boldsymbol{0}'_{\mathrm{e}}$ such that $\boldsymbol{a}$ is incomparable with any Δ^0_2 enumeration degree $\boldsymbol{0}_{\mathrm{e}} < \boldsymbol{c} < \boldsymbol{0}'_{\mathrm{e}}$ [Har11b]. Likewise these techniques can be applied via a finite injury proof to show the existence of a low$_2$ (i.e. $\boldsymbol{c}'' = \boldsymbol{0}''_{\mathrm{e}}$) noncuppable enumeration degree $\boldsymbol{c}$, thus yielding an easy constructive version—in the special case of the low$_2$ enumeration degrees—of

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Giorgi *et al*'s [GSY] proof that below every nonlow total Σ^0_2 enumeration degree $\boldsymbol{b}$ there exists a noncuppable enumeration degree.

The notion of jump uniformity can also be extended to to show that, for any good approximable set X

$$\mathit{InfSet}(X) \quad \equiv_{\mathrm{e}} \quad J^2_X \tag{1.2}$$

where $\mathit{InfSet}(X) =_{\mathrm{def}} \{\, e \mid \Phi^X_e \text{ is infinite} \,\}$ and J^2_X denotes the double enumeration jump of X.

Note 1.4. In fact $J^2_X \leq_{\mathrm{e}} \mathit{InfSet}(X)$ provided that $deg_{\mathrm{e}}(X)$ is good whereas, for *any* set X, $\mathit{InfSet}(X) \leq_{\mathrm{e}} J^2_X$.

The importance of this is that it gives us a more general methodology for the construction of a good—for example Σ^0_2—enumeration degree $\boldsymbol{a}$ such that $\boldsymbol{a}'$ lies in a given interval. Specifically it was these techniques that were used to show that, for every enumeration degree $\boldsymbol{b} \leq \mathbf{0}'_{\mathrm{e}}$ there exists a noncuppable degree $\mathbf{0}_{\mathrm{e}} < \boldsymbol{a} < \mathbf{0}'_{\mathrm{e}}$ such that $\boldsymbol{b}' \leq \boldsymbol{a}'$ and $\boldsymbol{a}'' \leq \boldsymbol{b}''$ [Har11c].

Now, noting firstly that if $\boldsymbol{a} < \mathbf{0}'_{\mathrm{e}}$ is noncuppable then $\boldsymbol{a}$ is properly downward Σ^0_2 (i.e. every $\mathbf{0}_{\mathrm{e}} < \boldsymbol{d} \leq \boldsymbol{a}$ is properly Σ^0_2) and that this also implies that $\boldsymbol{a}$ is quasiminimal (i.e. bounds no nonzero total degree) we are naturally led to the question—given the ubiquity of the downwards properly Σ^0_2 degrees—of whether the distribution of the Δ^0_2 quasiminimal degrees has similar characteristics. In particular we can ask whether there exists Δ^0_2 enumeration degree $\mathbf{0}_{\mathrm{e}} < \boldsymbol{a} < \mathbf{0}'_{\mathrm{e}}$ such that $\boldsymbol{a}$ is incomparable with every total degree $\mathbf{0}_{\mathrm{e}} < \boldsymbol{c} < \mathbf{0}'_{\mathrm{e}}$. However one half of this question is refuted in [ACK03] by the proof that there exists, for every Δ^0_2 enumeration degree $\boldsymbol{a} < \mathbf{0}'_{\mathrm{e}}$, a total degree $\boldsymbol{a} \leq \boldsymbol{c} < \mathbf{0}'_{\mathrm{e}}$. Hence only downward incomparability—i.e. quasiminimality—applies in the case of the Δ^0_2 enumeration degrees, so that the main question here is whether there exist Δ^0_2 quasiminimal enumeration degrees that are nonlow—since every quasiminimal low (i.e. $\boldsymbol{c}' = \mathbf{0}'_{\mathrm{e}}$) degree $\boldsymbol{c}$ is Δ^0_2. This question is addressed in [Har11a] where jump uniformity techniques are again employed—relative to $\mathbf{0}'_{\mathrm{e}}$—to build a quasiminimal Δ^0_2 enumeration degree $\boldsymbol{a} < \mathbf{0}'_{\mathrm{e}}$ which is high.

Jump uniformity methods also provide a means of studying exactly where goodness breaks down in the arithmetical hierarchy. It can be deduced from the density of the good enumeration degrees [LS92] and Calhoun and Slaman's proof [CS96] of the nondensity of the Π^0_2 enumeration degrees that there exists a bad Π^0_2 degree $\boldsymbol{a}$ such that $\boldsymbol{a}' \leq \mathbf{0}''_{\mathrm{e}}$. With this in mind, consider any Δ^0_2 enumeration degree $\boldsymbol{c}$. Then $\boldsymbol{c}$ contains a set C such that both C and $\overline{C}$ are Σ^0_2 and so both sets are good approximable. Hence the Π^0_2 degree $deg_{\mathrm{e}}(\overline{C})$ is good. From this point of view—given that all low sets are Δ^0_2—a tight bound on the breakdown of goodness can be displayed by showing the existence of a Σ^0_2 set X of low_2 jump complexity such that $\boldsymbol{y} = deg_{\mathrm{e}}(\overline{X})$ is bad. (Note here that the low_2-ness of X also implies that $\boldsymbol{y}' \leq \mathbf{0}''_{\mathrm{e}}$.) This result is achieved by constructing $\overline{X}$ via a $\Pi^{0,\mathcal{K}}_1$ approximation (i.e. using $\mathcal{K}$ as oracle) while ensuring that $\overline{X}$ is not jump uniform—so that $\boldsymbol{y} = deg_{\mathrm{e}}(\overline{X})$ is bad—and, at the same time, ensuring that $\mathit{InfSet}(X) \in \mathbf{0}''_{\mathrm{e}}$—which implies that $\boldsymbol{x}'' = \mathbf{0}''_{\mathrm{e}}$ using the fact that $\boldsymbol{x} =_{\mathrm{def}} deg_{\mathrm{e}}(X)$ is good, since X is Σ^0_2 [Har11a].

The main aim of the talk will be to present the fundamental ideas behind these results. I will conclude by describing a notion of *double jump uniformity* which applies in the Σ_2^0 enumeration degrees, and also by explaining the latter's significance relative to open problems in the study of the distribution of the properly Σ_2^0 enueration degrees.

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