# On a Hierarchy of Pluscupping Degrees

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# Basics definitions

- Noncuppable degrees
- Pluscupping degrees
- Minimal pairs and cappable degrees

# Cappable degrees - Some facts

### Theorem (Ambos-Spies, Jockusch, Shore and Soare, 1984)

*For a c.e. degree* $\mathbf{a}$*, the following are equivalent:*

- $\mathbf{a}$ *is noncappable;*
- $\mathbf{a}$ *is low-cuppable;*
- $\mathbf{a}$ *is promptly simple.*

The cappable degrees form an ideal of $\mathcal{R}$, denoted as $M$, and hence we can consider the quotient structure $\mathcal{R}/M$, and consider Shoenfield conjecture in this structure.

- Every nonzero c.e. degree bounds a cappable degree. (Exercise)

# Locally noncappable degrees

## Definition (Seetapun)

A c.e. degree $\mathbf{a}$ is locally noncappable if there is a c.e. degree $\mathbf{c} > \mathbf{a}$ such that no c.e. degree below $\mathbf{c}$ can form a minimal pair with $\mathbf{a}$.

Observation: If $\mathbf{a}$ is locally noncappable witnessed by $\mathbf{c} > \mathbf{a}$, and $\mathbf{a}$ bounding no minimal pairs, then $\mathbf{c}$ also bounds no minimal pairs.

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To see this, let $\mathbf{d}, \mathbf{e}$ be any two nonzero c.e. degrees below $\mathbf{c}$, forming a minimal pair. Then there are nonzero c.e. degrees $\mathbf{f} < \mathbf{a}, \mathbf{d}$ and $\mathbf{g} < \mathbf{a}, \mathbf{e}$, and $\mathbf{f}$ and $\mathbf{g}$ form a minimal pair, contradicting the assumption that $\mathbf{a}$ bounds no minimal pairs.

## Theorem (Seetapun)

*Each nonzero c.e. degree* $\mathbf{a}$ *is locally noncappable. So there are no maximal nonbound degrees.*

# Highness and Nonboundings

Observation: If $\mathbf{a}$ is locally noncappable witnessed by $\mathbf{c} > \mathbf{a}$, and $\mathbf{a}$ is a pluscupping degree (bounds no nonzero noncuppable degrees), then $\mathbf{c}$ is also pluscupping. Therefore, by Seetapun's result, there are no maximal pluscupping degrees.

- (Downey, Lempp and Shore) Nonbounding degrees can be $\text{high}_2$.
- (Li) Plus-cupping degrees can be $\text{high}_2$.

But they cannot be high.

### Theorem (Stephan and Wu)

$\mathbf{c}$ *is Seetapun's result can be high*$_2$*.*

This result is fairly strong, as it implies those results mentioned above immediately.

### Theorem (Fang, Wang and Wu)

*For any nonzero c.e. degree* $\mathbf{a}$*, there are c.e. degrees* $\mathbf{c}, \mathbf{e}$ *above* $\mathbf{a}$ *witnessing that* $\mathbf{a}$ *is locally noncappable and the* $\mathbf{c} \vee \mathbf{e}$ *is high.*

# A hierarchy of cuppable degrees

$LC_n$ denotes a subclass of cuppable degrees, each of which can be cupped to $\mathbf{0}'$ by a low$_n$ degree. So $LC_1$ is exactly the class of noncappable degrees, and

$$LC_1 \subseteq LC_2 \subseteq \cdots \subseteq LC_n \subseteq \cdots \subseteq CUP.$$

## Theorem (Li, Wu and Zhang)

$$LC_1 \subset LC_2.$$

Idea: construct a cappable and also low$_2$-cuppable degree.

# CUP is bigger

**Theorem (Greenberg, Ng and Wu)**

$$\bigcup_n LC_n \subset CUP.$$

Requirements:

- $A$ cuppable;
- If $A \oplus V$ computes $K$, then $V$ is high.

# Make $A$ cuppable

Construct c.e. sets $E, F$ and a p.c. functional $\Gamma$ such that

$$K = \Gamma^{A,E}$$

and $E$ incomplete:

$$F \neq \Phi^E$$

for any p.c. functional $\Phi$.

Note the part of putting numbers into $A$.

# Making $A$ only high-cuppable

We construct a c.e. set $P$ such that

$\mathcal{Q}_e$: $P = \Phi_e^{A,V_e} \quad \Rightarrow$ there is a p.c. functional $\Delta_e$ such that

$$TOT(i) = \lim_x \Delta_e^{V_e}(i,x)$$

for each $i$.

Idea of satisfying $\mathcal{Q}_e$: for each $i$, we try to satisfy the substrategy

$\mathcal{T}_{e,i}$: $TOT(i) = \lim_x \Delta_e^{V_e}(i,x)$.

# Make it high

Choose a high noncuppable degree $\mathbf{h}$, and consider $\mathbf{a} \vee \mathbf{h}$.

- high and only-high-cuppable.
- bounds noncuppable degrees.

Is it possible to have a plus-cupping and also only-high cuppable, degree?

# Main Result

## Theorem (Wang and Wu)

*These exists a pluscupping degree which is also only-high-cuppable.*

It refutes a claim of Li and W. Wang:

Let $PC_n$ be the collection of pluscupping degrees, such that all the nonzero degrees below them are low$_n$-cuppable.

- $PC_n = \emptyset$;
- $PC_1 \subseteq PC_2$;
- Li and W. Wang's claim: $PC_3 = PC$, which is not true by our theorem.

*Thanks!*