Cupping and Diamond Embeddings: A Unifying Approach

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Basics of cupping in c.e. degrees

- Computably enumerable sets and degrees
- Cuppable degrees and Noncuppable degrees - definition
- Yates, Cooper and Harrington - noncuppable degree construction

- Harrington's nonbounding Theorem - plus-cupping degrees

Theorem

There is a nonzero c.e. degree such that every nonzero c.e. degree below it is cuppable.

- Slaman's Cupping Theorem

Theorem

There are incomplete c.e. degrees $\mathbf{a}$ *and* $\mathbf{c}$ *such that any nonzero c.e. degree below* $\mathbf{a}$*, but not below* $\mathbf{c}$*, cups* $\mathbf{c}$ *to* $\mathbf{0}'$*.*

How about these two degrees?

Theorem (Li, Wu and Yang)

There are two cuppable c.e. degrees $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{0}'$ *is the only one c.e. degree cupping both of them to* $\mathbf{0}'$*.*

In other words, in the quotient structure $\mathbf{R}/NCup$, there exists a minimal pair.

Difference of c.e. sets

- A set A is d.c.e. if there are c.e. sets B and C such that $A = B - C$.
- Effective approximations and generalizations - n-c.e. sets, ω-c.e. sets, α-c.e. sets

  Ershov hierarchy
- A Turing degree is d.c.e. if it contains a d.c.e. set.

Cooper and Lachlan - 70's

- *There are proper d.c.e. degrees.* (Cooper)
- *The d.c.e. degrees are downwards dense.* (Lachlan)
- *The low_2 d.c.e. degrees are dense.* (Cooper)

Arslanov and Downey - 80's

- *Every nonzero d.c.e. degree is cuppable to* $\mathbf{0}'$ *by a d.c.e. degree.* (Arslanov)
- *The diamond lattice can be embedded into the d.c.e. degrees preserving 0 and 1.* (Downey)

Arslanov - basic ideas

Given a nonzero c.e. degree $\mathbf{a}$, construct an incomplete d.c.e. degree $\mathbf{d}$ such that $\mathbf{a} \vee \mathbf{d} = \mathbf{0}'$.

A direct conflict is between coding K into $A \oplus D$ and making D incomplete.

$$E \neq \Phi^D.$$

- *Arslanov's construction can be given by applying the so-called threshold strategy.*
- *Note that A is given as an incomputable set.*

Downey - basic ideas

There are nonzero d.c.e. degrees $\mathbf{c}$ *and* $\mathbf{d}$ *such that* $\mathbf{a} \vee \mathbf{d} = \mathbf{0}'$ *and* $\mathbf{a} \wedge \mathbf{d} = \mathbf{0}$.

A direct conflict is between coding K into $C \oplus D$ and making C and D forming a minimal pair.

- *An Alternative Approach:*

  There are nonzero c.e. degrees $\mathbf{c}$ and $\mathbf{a}$ and a d.c.e. degree $\mathbf{d} > \mathbf{a}$ such that $\mathbf{c} \vee \mathbf{d} = \mathbf{0}'$ and $\mathbf{c} \wedge \mathbf{a} = \mathbf{0}$ and all the c.e. degrees below $\mathbf{d}$ are also below $\mathbf{a}$.

- $\mathbf{d}$ above is said to be isolated by $\mathbf{a}$, according to Cooper and Yi.
- Note that $\{\mathbf{0}, \mathbf{c}, \mathbf{d}, \mathbf{0}'\}$ is a diamond embedding wanted by Downey.
- *The cupping and the capping are separated into two different steps.*

Theorem (Downey, Li and Wu)

For any given nonzero cappable degree $\mathbf{c}$*, there are a d.c.e. degrees* $\mathbf{d}$ *and a c.e. degree* $\mathbf{a} < \mathbf{d}$*, isolating* $\mathbf{d}$*, such that* $\mathbf{c} \vee \mathbf{d} = \mathbf{0}'$ *and* $\mathbf{c} \wedge \mathbf{a} = \mathbf{0}$*.*

As a corollary, every cappable c.e. degree is complementable in the d.c.e. degrees.

Note that this implies both Arslanov's and Downey's results mentioned above.

CHLLS - 90's

- *The d.c.e. degrees are not dense. In particular, there exists a maximal incomplete d.c.e. degree.* (Cooper, Harrington, Lachlan, Lempp and Soare)

  Such incomplete maximal d.c.e. degrees, $\mathbf{d}$ say, have nice cupping properties: $\mathbf{d}$ cups all the c.e. degrees not below it to $\mathbf{0}'$.

- A d.c.e. degree is said to have almost universal cupping property , if it cups all the c.e. degrees not below it to $\mathbf{0}'$.

- Such a d.c.e. degree can be isolated. That is, there exist a d.c.e. degree $\mathbf{d}$ and a c.e. degree $\mathbf{a} < \mathbf{d}$ such that all the c.e. degrees that cannot be cupped to $\mathbf{0}'$ by $\mathbf{d}$ are less than or equal to $\mathbf{a}$. (Liu and Wu, CiE 2010)

Cupping c.e. degrees to $\mathbf{0}'$

- There exists an incomplete ω-c.e. degree which cups each nonzero c.e. degree to $\mathbf{0}'$. (Li, Song and Wu)

- No single n-c.e. degree can take this job.

Theorem (Li and Yi)

There are two d.c.e. degrees $\mathbf{d}_1, \mathbf{d}_2$ such that any nonzero c.e. degree cups at least one of these two d.c.e. degrees to $\mathbf{0}'$.

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This theorem is strong enough, which also implies Arslanov's and Downey's results immediately.

A Unifying Approach

Theorem (Fang, Liu and Wu)

For any given nonzero cappable degree $\mathbf{c}$*, there are a d.c.e. degrees* $\mathbf{d}$ *and a c.e. degree* $\mathbf{a} < \mathbf{d}$*, isolating* $\mathbf{d}$*, such that* $\mathbf{c} \wedge \mathbf{a} = \mathbf{0}$ *and* $\mathbf{d}$ *has the almost-universal cupping property.*

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How can we obtain Li-Yi's result?

Apply our result twice, and we can have two d.c.e. degrees with almost-universal cupping property and forming a minimal pair in the d.c.e. degrees.

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Recall that the construction of incomplete maximal d.c.e. degrees is hard.

Enumeration degrees

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Soskova has some progress on this aspect, about cupping Δ_2^0 degrees to $\mathbf{0}'$.

Thanks!