Conservative Extensions of Structures

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Enumerations of Structures

In this talk we shall study a relation between structures $\mathfrak{A}$ and $\mathfrak{B}$, possibly with different signatures and $|\mathfrak{A}| \subseteq |\mathfrak{B}|$.
The structures will have the form $\mathfrak{A} = (A; P_1, \ldots, P_k, =)$, where A is an arbitrary countable set.

- Let f be an injective mapping onto A and $\mathrm{Dom}(f) \subseteq \mathbb{N}$.
- For $X \subseteq A^m$, we denote the pullback of X under f as $f^{-1}(X) = \{\langle x_1 \ldots x_m \rangle : (f(x_1), \ldots, f(x_m)) \in X\}$.
- The pullback of $\mathfrak{A}$ under f is the set $f^{-1}(\mathfrak{A}) = f^{-1}(P_1) \oplus \cdots \oplus f^{-1}(P_k) \oplus f^{-1}(=)$.

Informally speaking, you can think of $f^{-1}(\mathfrak{A})$ as the atomic diagram of the copy of $\mathfrak{A}$ under f.

Enumerations of Structures

Definition

We call an enumeration of $\mathfrak{A}$ the pair $\alpha = (\mathrm{f}_\alpha, \mathrm{R}_\alpha)$, where

(i) f_α is an injective mapping onto A and $\mathrm{Dom}(\mathrm{f}_\alpha) \subseteq \mathbb{N}$,

(ii) $\mathrm{R}_\alpha \subseteq \mathbb{N}$ and $\mathrm{f}_\alpha^{-1}(\mathfrak{A}) \leq_{\mathrm{T}} \mathrm{R}_\alpha$.

For an enumeration α, we denote $\alpha^{(\mathrm{n})} = (\mathrm{f}_\alpha, \mathrm{R}_\alpha^{(\mathrm{n})})$.

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For an enumeration α, we denote $\alpha^{(\mathrm{n})} = (\mathrm{f}_\alpha, \mathrm{R}_\alpha^{(\mathrm{n})})$.

Definition (Comparing two enumerations)

Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $|\mathfrak{A}| \subseteq |\mathfrak{B}|$, $\alpha = (\mathrm{f}_\alpha, \mathrm{R}_\alpha)$ and $\beta = (\mathrm{f}_\beta, \mathrm{R}_\beta)$ are enumerations of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. We denote $\alpha \leq \beta$ if:

(i) $\mathrm{R}_\alpha \leq_{\mathrm{T}} \mathrm{R}_\beta$, and

(ii) $\mathrm{E}(\mathrm{f}_\alpha, \mathrm{f}_\beta)$ is c.e. in R_β, where

$$\mathrm{E}(\mathrm{f}_\alpha, \mathrm{f}_\beta) = \{\langle \mathrm{x}, \mathrm{y} \rangle \mid \mathrm{x} \in \mathrm{Dom}(\mathrm{f}_\alpha) \;\&\; \mathrm{y} \in \mathrm{Dom}(\mathrm{f}_\beta) \;\&\; \mathrm{f}_\alpha(\mathrm{x}) = \mathrm{f}_\beta(\mathrm{y})\}.$$

Comparing Structures

Definition

Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures, possibly with different signatures and $|\mathfrak{A}| \subseteq |\mathfrak{B}|$.

(i) $\mathfrak{A} \Rightarrow \mathfrak{B}$, if for every enumeration β of $\mathfrak{B}$, there exists an enumeration α of $\mathfrak{A}$, such that $\alpha \leq \beta$.

(ii) $\mathfrak{A} \Leftarrow \mathfrak{B}$, if for every enumeration α of $\mathfrak{A}$, there exists an enumeration β of $\mathfrak{B}$, such that $\beta \leq \alpha$.

(iii) $\mathfrak{A} \Leftrightarrow \mathfrak{B}$, if $\mathfrak{A} \Rightarrow \mathfrak{B}$ and $\mathfrak{A} \Leftarrow \mathfrak{B}$.

Comparing Structures

We generalize the previous definition for arbitrary $\mathrm{k}, \mathrm{n} \in \mathbb{N}$.

Definition

Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures, possibly with different signatures and $|\mathfrak{A}| \subseteq |\mathfrak{B}|$.

(i) $\mathfrak{A} \Rightarrow^{\mathrm{k}}_{\mathrm{n}} \mathfrak{B}$, if for every enumeration β of $\mathfrak{B}$, there exists an enumeration α of $\mathfrak{A}$, such that $\alpha^{(\mathrm{k})} \leq \beta^{(\mathrm{n})}$.

(ii) $\mathfrak{A} \Leftarrow^{\mathrm{k}}_{\mathrm{n}} \mathfrak{B}$, if for every enumeration α of $\mathfrak{A}$, there exists an enumeration β of $\mathfrak{B}$, such that $\beta^{(\mathrm{k})} \leq \alpha^{(\mathrm{n})}$.

(iii) $\mathfrak{A} \Leftrightarrow^{\mathrm{k}}_{\mathrm{n}} \mathfrak{B}$, if $\mathfrak{A} \Rightarrow^{\mathrm{k}}_{\mathrm{n}} \mathfrak{B}$ and $\mathfrak{A} \Leftarrow^{\mathrm{n}}_{\mathrm{k}} \mathfrak{B}$.
In this case we say that $\mathfrak{B}$ is (k, n)-conservative extension of $\mathfrak{A}$.

Characterization

Definition (Computably infinitary formulas)

- Σ^c_0 and Π^c_0 formulas are quantifier-free formulas with a finite number of conjunctions and disjunctions.
- Σ^c_{n+1} formulas have the form $\varphi(\overline{x}) \equiv \bigvee_{i \in W_e} \exists \overline{y} \psi_i(\overline{x}, \overline{y})$, where W_e is a c.e. set with index e and ψ_i are Π^c_n formulas.
- Π^c_{n+1} formulas are negations of Σ^c_{n+1} formulas.

Definition

We say that a set $X \subseteq A$ is definable in $\mathfrak{A}$ with a Σ^c_n formula, denoted with $X \in \Sigma^c_n(\mathfrak{A})$, if there exists a tuple of constants $\overline{c}$ in $|\mathfrak{A}|$ and a Σ^c_n formula $\psi(x, \overline{y})$ such that

$$x \in X \iff \mathfrak{A} \models \psi(x, \overline{c}).$$

Characterization

Theorem (Ash-Knight-Manasse-Slaman, Chisholm)

Let $\mathfrak{A}$ be a countable structure. For every set $X \subseteq A$,

$$X \in \Sigma^c_{n+1}(\mathfrak{A}) \iff (\forall \alpha \text{ enumeration of } \mathfrak{A})[f^{-1}(X) \text{ is c.e. in } R^{(n)}_\alpha].$$

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The following theorem gives the reason for our choice of notation.

Theorem
Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures and $|\mathfrak{A}| \subseteq |\mathfrak{B}|$. Then for every $k, n \in \mathbb{N}$,

(i) if $\mathfrak{A} \Rightarrow^k_n \mathfrak{B}$, then $(\forall X \subseteq A)[X \in \Sigma^c_{k+1}(\mathfrak{A}) \to X \in \Sigma^c_{n+1}(\mathfrak{B})]$;

(ii) if $\mathfrak{A} \Leftarrow^n_k \mathfrak{B}$, then $(\forall X \subseteq A)[X \in \Sigma^c_{n+1}(\mathfrak{B}) \to X \in \Sigma^c_{k+1}(\mathfrak{A})]$;

(iii) if $\mathfrak{A} \Leftrightarrow^k_n \mathfrak{B}$, then $(\forall X \subseteq A)[X \in \Sigma^c_{k+1}(\mathfrak{A}) \leftrightarrow X \in \Sigma^c_{n+1}(\mathfrak{B})]$.

In the general case, we do not have the reverse implications.

Moschovakis' Extension

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Let $A_0 = A \cup \{\bar{0}\}$, $\bar{0} \notin A$. Let Π be a function, which gives ordered pairs, such that no element of A_0 is an ordered pair.
$A^\star$ is the least set containing A_0 and closed under Π.

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- We can encode the natural numbers in $A^\star$. Let $0^\star = \bar{0}$ and $(n+1)^\star = \Pi(\bar{0}, n^\star)$. We denote them by $\mathbb{N}^\star$.

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Let $A_0 = A \cup \{\bar{0}\}$, $\bar{0} \notin A$. Let $\sqcap$ be a function, which gives ordered pairs, such that no element of A_0 is an ordered pair.
$A^\star$ is the least set containing A_0 and closed under $\sqcap$.

- We can encode the natural numbers in $A^\star$. Let $0^\star = \bar{0}$ and $(n+1)^\star = \sqcap(\bar{0}, n^\star)$. We denote them by $\mathbb{N}^\star$.
- Moreover, we can encode finite sequences of elements of $A^\star$. For every $t_1, \ldots, t_{n+1} \in A^\star$, let $\sqcap_1(t_1) = t_1$ and $\sqcap_{n+1}(t_1, \ldots, t_{n+1}) = \sqcap(t_1, \sqcap_n(t_2, \ldots, t_{n+1}))$

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- We can encode the natural numbers in $A^\star$. Let $0^\star = \bar{0}$ and $(n+1)^\star = \Pi(\bar{0}, n^\star)$. We denote them by $\mathbb{N}^\star$.
- Moreover, we can encode finite sequences of elements of $A^\star$. For every $t_1, \dots, t_{n+1} \in A^\star$, let $\Pi_1(t_1) = t_1$ and $\Pi_{n+1}(t_1, \dots, t_{n+1}) = \Pi(t_1, \Pi_n(t_2, \dots, t_{n+1}))$
- $A^\star$ resembles $\mathbb{HF}(A)$.

Moschovakis' Extension

The next step is to define functions for decoding. Let L and R be function onto $A^\star$ such that:

$$L(\bar{0}) = R(\bar{0}) = 0^\star;$$
$$(\forall t \in A)[L(t) = R(t) = 1^\star];$$
$$(\forall s, t \in A^\star)[L(\Pi(s,t)) = s \;\&\; R(\Pi(s,t)) = t].$$

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Definition (Moschovakis' Extension)

Let a structure $\mathfrak{A}$ be given. For every predicate P_i of $\mathfrak{A}$, we define the unary predicate $P_i^\star$ on $A^\star$ such that:

$$P_i^\star(t) \iff (\exists a_1, \ldots, a_{n_i} \in A)[t = \Pi_{n_i}(a_1, \ldots, a_{n_i}) \& P_i(a_1, \ldots, a_{n_i})].$$

Moschovakis' extension of $\mathfrak{A}$ is the structures

$$\mathfrak{A}^\star = (A^\star; A_0, P_1^\star, \ldots, P_k^\star, G_\Pi, G_L, G_R, =),$$

where G_Π, G_L and G_R are the graphs of Π, L и R.

Characterization

Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $|\mathfrak{A}| \subseteq |\mathfrak{B}|$ and suppose we use the same coding for all structures.

Proposition

$\mathfrak{A} \Leftrightarrow_{\mathrm{n}}^{\mathrm{k}} \mathfrak{B}$ if and only if $\mathfrak{A}^\star \Leftrightarrow_{\mathrm{n}}^{\mathrm{k}} \mathfrak{B}^\star$. In particular, $\mathfrak{A} \Leftrightarrow_{\mathrm{n}}^{\mathrm{n}} \mathfrak{A}^\star$.

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Theorem

Suppose that $|\mathfrak{A}^\star| \subseteq |\mathfrak{B}|$ and

$$(\forall X \subseteq A^\star)[X \in \Sigma^c_{k+1}(\mathfrak{A}^\star) \to X \in \Sigma^c_{n+1}(\mathfrak{B})].$$

Then $\mathfrak{A} \Rightarrow_n^k \mathfrak{B}$.

Corollary

$\mathfrak{A} \Rightarrow_n^k \mathfrak{B}$ iff $(\forall X \subseteq A^\star)[X \in \Sigma^c_{k+1}(\mathfrak{A}^\star) \to X \in \Sigma^c_{n+1}(\mathfrak{B}^\star)]$.

Applications. Degree Spectra

Definition

- The Turing degree spectrum of $\mathfrak{A}$ is the set $\mathrm{DS}(\mathfrak{A}) = \{\mathrm{d_T}(\mathrm{R}_\alpha) \mid \alpha \text{ is an enumeration of } \mathfrak{A}\}$.
- The k-th Turing jump spectrum of $\mathfrak{A}$ is the set $\mathrm{DS_k}(\mathfrak{A}) = \{\mathrm{d_T}(\mathrm{R}_\alpha^{(\mathrm{k})}) \mid \alpha \text{ is an enumeration of } \mathfrak{A}\}$.

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Proposition

Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $|\mathfrak{A}| \subseteq |\mathfrak{B}|$.

- If $\mathfrak{A} \Rightarrow^{\mathrm{k}}_{\mathrm{n}} \mathfrak{B}$, then $\mathrm{DS_n}(\mathfrak{B}) \subseteq \mathrm{DS_k}(\mathfrak{A})$;
- If $\mathfrak{A} \Leftarrow^{\mathrm{n}}_{\mathrm{k}} \mathfrak{B}$, then $\mathrm{DS_k}(\mathfrak{A}) \subseteq \mathrm{DS_n}(\mathfrak{B})$;
- If $\mathfrak{A} \Leftrightarrow^{\mathrm{k}}_{\mathrm{n}} \mathfrak{B}$, then $\mathrm{DS_k}(\mathfrak{A}) = \mathrm{DS_n}(\mathfrak{B})$;

In the general case, we do not have the reverse implications.

Applications. The Jump Structure

Let a structure $\mathfrak{A}$ be given and P be a relation in $\mathfrak{A}$, $\tau \in \mathrm{Fin}(A)$.

$$\tau^{-1}(P)(u) \downarrow= 1 \leftrightarrow (\exists x_1, \dots, x_n \in \mathrm{Dom}(\tau))[u = \langle x_1, \dots, x_n \rangle \; \& \; (\tau(x_1), \dots, \tau(x_n)) \in P],$$

$$\tau^{-1}(P)(u) \downarrow= 0 \leftrightarrow (\exists x_1, \dots, x_n \in \mathrm{Dom}(\tau))[u = \langle x_1, \dots, x_n \rangle \; \& \; (\tau(x_1), \dots, \tau(x_n)) \notin P].$$

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- $\tau \Vdash_{n+1} F_e(x) \leftrightarrow \ (\exists \delta \in \mathrm{Fin}(\{0,1\}))[x \in W_e^{\delta} \ \&$ $(\forall z \in \mathrm{Dom}(\delta))[(\delta(z) = 1 \ \& \ \tau \Vdash_n F_z(z)) \vee$ $(\delta(z) = 0 \ \& \ \tau \Vdash_n \neg F_z(z))]]$,
- $\tau \Vdash_n \neg F_e(x) \leftrightarrow \ (\forall \rho \in \mathrm{Fin}(A))[\tau \subseteq \rho \rightarrow \rho \nVdash_n F_e(x)]$.

Applications. The Jump Structure

We define in some sense an analogue of the Kleene set.

$$K_n^{\mathfrak{A}} = \{\Pi_3(\delta^\star, e^\star, x^\star) \mid (\exists \tau \in \mathrm{Fin}(A))[\delta \subseteq \tau \,\&\, \tau \Vdash_n F_e(x)] \,\&\, e^\star, x^\star \in \mathbb{N}^\star\}$$

Definition

The n-th jump structure of $\mathfrak{A}$ is defined as:
$\mathfrak{A}^{(0)} = \mathfrak{A}$ and $\mathfrak{A}^{(n+1)} = (\mathfrak{A}^\star, K_n^{\mathfrak{A}})$.

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- Moreover, $(\forall X \subseteq A)[X \in \Sigma_{n+2}^c(\mathfrak{A}) \leftrightarrow X \in \Sigma_{n+1}^c(\mathfrak{A}')]$;

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- Moreover, $(\forall X \subseteq A)[X \in \Sigma_{n+2}^c(\mathfrak{A}) \leftrightarrow X \in \Sigma_{n+1}^c(\mathfrak{A}')]$;
- $K_n^{\mathfrak{A}} \in \Sigma_{n+1}^c(\mathfrak{A}^\star)$ and $A^\star \setminus K_n^{\mathfrak{A}} \notin \Sigma_{n+1}^c(\mathfrak{A}^\star)$.
- $\mathfrak{A}^{(n)} \Rightarrow_0^0 \mathfrak{A}^{(n+1)}$ and $\mathfrak{A}^{(n)} \not\Leftarrow_0^0 \mathfrak{A}^{(n+1)}$.
  Therefore, the sequence $\{\mathfrak{A}, \mathfrak{A}^{(1)}, \mathfrak{A}^{(2)}, \ldots, \mathfrak{A}^{(n)}, \ldots\}$ is strictly increasing under the relation $\Rightarrow_0^0$.

Applications. Marker's extension

For $\mathfrak{A} = (A; P_1, \ldots, P_k, =)$, the Marker's extension of $\mathfrak{A}$ is a structure of the form

$$\mathfrak{A}^{\exists\forall} = (A \cup \bigcup_{i=1}^{k} X_i \cup \bigcup_{i=1}^{k} Y_i; P_1^{\exists\forall}, \ldots, P_k^{\exists\forall}, X_1, \ldots, X_k, Y_1, \ldots, Y_k, =).$$

Among others, it has the following property:

$$\mathfrak{A} \models P_i(\bar{a}) \text{ if and only if } \mathfrak{A}^{\exists\forall} \models (\exists x \in X_i)(\forall y \in Y_i)[P_i^{\exists\forall}(\bar{a}, x, y)]$$

Theorem (A. Soskova - I. Soskov)

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures with $DS(\mathfrak{A}) \subseteq DS_1(\mathfrak{B})$.
Then there exists a structure $\mathfrak{C}$ such that
$DS(\mathfrak{C}) \subseteq DS(\mathfrak{B})$ and $DS_1(\mathfrak{C}) = DS(\mathfrak{A})$.

Applications. Marker's Extension

We give an analogue of their result. Let $\mathcal{O}_A = (A, =)$.

Proposition (Jump Inversion)

Let $\mathfrak{A}$ be a countable structure and $\mathcal{O}_A \Rightarrow_0^k \mathfrak{A}$ for some $k \in \mathbb{N}$. There exists a countable structure $\mathfrak{B}$ such that $\mathfrak{A} \Leftrightarrow_0^0 \mathfrak{B}^{(k)}$. In other words,

$$(\forall X \subseteq A)[X \in \Sigma_1^c(\mathfrak{A}) \leftrightarrow X \in \Sigma_1^c(\mathfrak{B}^{(k)})].$$

It is still open whether:

- We can relativize it, i.e. replace $\mathcal{O}_A \Rightarrow_0^k \mathfrak{A}$ with $\mathfrak{C} \Rightarrow_0^k \mathfrak{A}$ for arbitrary $\mathfrak{C}$.
- Lift the result to infinite ordinals.

БЛАГОДАРЯ!
(Thank you!)