Orbit Relation and Isomorphism Type for Computable Trees Under Predecessor

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How complicated is it for two nodes on such a tree to be interchangeable under an automorphism?

orbit relation

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claim

The orbit relation on a computable, finite-branching tree under predecessor is Π^0_2.

lemma 1

If $\mathcal{T}$ and $\mathcal{S}$ are finite-branching trees under predecessor which are isomorphic to each other, then every embedding of $\mathcal{T}$ into $\mathcal{S}$ must be surjective.

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Two nodes x and y on a computable, finite-branching tree under predecessor are in the same orbit iff for every node z with $meet(x, y) < z \leq x$ and every node w with $meet(x, y) < w \leq y$, the tree above z is isomorphic to the tree above w.

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where $p(X) \in \mathbb{Q}[X]$.

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where $p(X) \in \mathbb{Q}[X]$.

It is the effective version of the Theorem of the Primitive Element which allows us to do this.

the case for trees under predecessor

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These sentences must also be true in $\mathcal{T}$, and hence true in $\mathcal{S}$ by assumption. So each $\mathcal{T}_n$ embeds into $\mathcal{S}$.

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A similar argument shows that each $\mathcal{S}_n$ embeds into $\mathcal{T}$. Then by lemma 2 from before, $\mathcal{T} \cong \mathcal{S}$.

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So we can say that the isomorphism type of an algebraic field is completely determined by its *single-quantifier* Σ_1 theory.

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One needs arbitrarily many $\exists$ quantifiers to determine the isomorphism type of a finite-branching tree under predecessor.

another note

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The generalized orbit relation on pairs of *n*-tuples of elements of a computable, finite-branching tree under predecessor is Π^0_2.

The corresponding result for algebraic fields is no different from the result about the orbit relation on pairs of single elements, because of the Theorem of the Primitive Element.

Yet, even though we don't have such a theorem for finite-branching trees under predecessor, the generalized orbit relation is still Π^0_2.

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computable categoricity

Computable categoricity for finite-branching trees under predecessor is Π^0_4.

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Computable categoricity for algebraic fields is Π_4^0-complete.

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and we conjecture the following:

conjecture

Computable categoricity for finite-branching trees under predecessor is Π^0_4-complete.