The First-order Fragments of Second-order Theories

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Motivating Question

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There is an impression that using infinitary methods to draw conclusions about finite objects can obscure the real reason that those conclusions are true.

- *What strengths do infinitary principles/methods add to our ability to understand the finite sets, if any?*
- *Can we characterize this additional strength in purely finite terms?*

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- 2^ω *is compact.* For every infinite binary tree T, there is an infinite path through T.
- 2^ω *is not meager.* For every countable collection of dense open sets, there is a "generic" element in their intersection.
- 2^ω *supports a continuous, countably additive, probability measure.* For every sequence of open sets whose measures approach zero, there is a "random" element in the complement of their intersection.

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What are the number theoretic consequences of infinitary principles?

The answer depends on

- *Base Theory.*
  - Basic number theory, algebraic operations on strings, definition of computable functions by recursion.
  - The basic closure properties assumed of infinite sets, which for us will be that they are closed under relative computability.
- *The descriptions of the trees, the sequences of dense opens, or the null G_δ's.*

Subsystems of Second Order Arithmetic: RCA_0

Definition

A *model* $\mathfrak{M}$ *of second-order arithmetic* consists of a structure $\mathfrak{N}$ for first-order arithmetic, called the *numbers* of $\mathfrak{M}$, and a collection of subsets of $\mathfrak{N}$, called the *reals* of $\mathfrak{M}$.

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Definition

RCA_0 is the second-order theory formalizing the following.

- P^-, the axioms for the nonnegative part of a discretely ordered ring.
- $I\Sigma_1$, for φ a Σ^0_1 predicate, if 0 is a solution to φ and the solutions to φ are closed under successor, then φ holds of all numbers.
- The reals are closed under join and relative Δ^0_1-definability.

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When we talk about definability, as we did above, parameters are allowed to appear in the definitions.

Nonstandard Models of Arithmetic

We will be giving semantic calculations of the number theoretic consequences of theories. Of necessity, we will exhibit non-standard models of first order arithmetic whose theories diverge from that of the natural numbers.

Base Theory: RCA_0

Observation

If $\mathfrak{N} \models P^- + I\Sigma_1$, then there is an $\mathfrak{M}$ obtained by adding reals to $\mathfrak{N}$ such that $\mathfrak{M} \models RCA_0$.

Proof.

The reals of $\mathfrak{M}$ are the sets which are Δ_1-definable in $\mathfrak{N}$. □

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Corollary

RCA_0 is conservative over $P^- + I\Sigma_1$ for arithmetic sentences, or even Π^1_1 sentences.

This means that any arithmetic sentence provable from RCA_0 is also provable from $P^- + I\Sigma_1$.

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Equivalent formulations of compactness for 2^ω.

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Typical applications of compactness. Exhibit consistent finite properties and conclude the existence of a limit which realizes all of them.

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- A finitely satisfiable collection of first order formulas is satisfiable.

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Compactness provides an existence principle which goes beyond the computable.

WKL_0, Weak Kőnig's Lemma

What arithmetic sentences can be proven by computable methods and applications of compactness?

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Definition

WKL_0 is RCA_0 augmented by the assertion that the reals are compact.

- If T is an unbounded tree of finite binary strings, then T has an infinite path.

The question above becomes, *What arithmetic sentences can be proven from WKL_0?*

WKL_0, Weak Kőnig's Lemma

Theorem (Harrington Conservation Theorem)

If a Π^1_1-sentence is provable from WKL_0, then it is provable from RCA_0.

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Proof.

Any countable model $\mathfrak{M}$ of RCA_0 can be extended to a model of WKL_0 by adding reals. Specifically, extend by forcing with unbounded binary trees in $\mathfrak{M}$. Conclude that if a Σ^1_1-sentence is consistent with RCA_0, then it is consistent with WKL_0. □

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In particular, simple appeals to compactness to prove number-theoretic statements can be replaced by effective inductions.

The Baire Category Theorem

The Principle of Category. The Baire Category Theorem asserts that the intersection of a countable collection of dense open subsets of 2^ω is nonempty. (2^ω is not *meager*.)

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Typical applications of the Baire Category Theorem. Specify a countable sequence of dense open sets and concludes that a *generic* point in their intersection has interesting properties.

- A continuous function that is nowhere differentiable
- A nonrecursive sequence G such that $G' \equiv_T G \oplus 0'$.
- The method of forcing in set theory.

Degrees of Genericity: 1-Generic

Recall, $G \in 2^\omega$ is *1-generic* if for every Σ^0_1 subset S of 2^ω, there is an ℓ such that either $G \upharpoonright \ell \in S$ or no τ extending $G \upharpoonright \ell$ is in S.

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Definition

1-G is RCA_0 augmented by the assertion that for every real X there is a G which is 1-random relative to X.

Degrees of Genericity: 1-Generic

Theorem (Brown and Simpson, 1993)

If a Π^1_1-sentence is provable from 1-G, then it is provable from RCA_0.

Proof.

Any countable model $\mathfrak{M}$ of RCA_0 can be extended to a model of 1-G by adding reals. Application of Σ_1-induction to iteratively make Σ^0_1 sentences true about G ensures $I\Sigma_1(G)$. □

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- *In particular, simple applications of genericity to prove number-theoretic statements can be eliminated.*
- *However, the existence of more fully generic sequences does have arithmetic content.*

The Kirby and Paris Hierarchy

$I\Sigma_n$. "Any Σ_n set of numbers including 0 and closed under successor is the set of all numbers."

Equivalences (Paris and Kirby, 1977):

- $I\Pi_n$
- $L\Sigma_n$, the least number principle for Σ_n sets
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$B\Sigma_n$. "If for all $x < a$ there is a y satisfying a Σ_n formula, then there is a bound b such that for all $x < a$ there is a $y < b$ satisfying the Σ_n formula."

Equivalences:

- (Kirby and Paris, 1977) $B\Pi_{n-1}$
- (Slaman, 2004) $I\Delta_n$ (for $n = 1$, proof requires exponentiation)

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$B\Sigma_n$ sound bite:

- The numbers form a Σ_n-regular cardinal.

The Kirby and Paris Hierarchy

Theorem (Kirby and Paris, 1977)

For $n > 0$ and working over $P^- + I\Sigma_0$, the induction and bounding principles are ordered by the following strict inclusions, written in terms of logical consequence.

$$\cdots \subset B\Sigma_n \subset I\Sigma_n \subset B\Sigma_{n+1} \subset \cdots$$

Degrees of Genericity: Π^0_1-Generic

Definition

Π^0_1-G is RCA_0 augmented by the assertion that for any uniformly Π^0_1 collection of sets D_i, each of which is dense in $2^{<\omega}$, there is a G such that $\forall i \exists \boldsymbol{\ell} (G \upharpoonright \boldsymbol{\ell} \in D_i)$.

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Theorem (Hirschfeldt, Shore, and Slaman, 2009)

$RCA_0 + B\Sigma_2 \vdash (\Pi^0_1\text{-}G \to I\Sigma_2)$

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Theorem (Hirschfeldt, Shore, and Slaman, 2009)

$RCA_0 + B\Sigma_2 \vdash (\Pi^0_1\text{-}G \rightarrow I\Sigma_2)$

For any countable model $\mathfrak{M}$ of $RCA_0 + B\Sigma_2$, there are predicates G which are arbitrarily $\mathfrak{M}$-generic. However, generic predicates fail to preserve $\mathfrak{M}$'s number-theoretic properties by compressing the Kirby-Paris Hierarchy.

Measure

Lebesgue Measure. 2^ω supports a continuous regular probability measure λ, in which $\lambda([\sigma]) = \frac{1}{2^{|\sigma|}}$.

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Typical Measure Theoretic Applications.

- A montone function $f : [0,1] \to \mathbb{R}$ is differentiable almost everywhere, i.e. differentiable at a random real.
- Monte Carlo methods.
- Randomized structures, such as the random graph.
- Ergodic theory and its applications.

Degree of Randomness: 1-Random

Recall, $R \in 2^\omega$ is 1-random iff R does not belong to any effectively-presented null G_δ set. (G_δ means countable intersection of open sets.)

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Theorem (Corollary to Harrington's Theorem)

If an arithmetic sentence φ can be proven from RCA_0 and the assertion that for every set X there is a set R which is 1-random relative to X, then φ can be proven from $P^- + I\Sigma_1$.

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If an arithmetic sentence φ can be proven from RCA_0 and the assertion that for every set X there is a set R which is 1-random relative to X, then φ can be proven from $P^- + I\Sigma_1$.

Proof.

RCA_0 proves that there is an unbounded tree for which any unbounded path is 1-random. Thus, WKL_0 proves the existence of 1-random reals. Use the Harrington conservation theorem. □

Degree of Randomness: 2-Random

Definition

A sequence R is 2-random relative to X iff R is 1-random relative to X'.

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R is 2-random relative to X' iff for every uniformly-recursive-in-X' sequence of trees T_n for which $\lambda(T_n) \geq 1 - 1/2^n$, there is a k such that R is a path in T_k.

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A sequence R is 2-random relative to X iff R is 1-random relative to X'.

R is 2-random relative to X' iff for every uniformly-recursive-in-X' sequence of trees T_n for which $\lambda(T_n) \geq 1 - 1/2^n$, there is a k such that R is a path in T_k.

- Such a sequence of trees is called a Martin-Löf test relative to X'.
- Here, the measure of a tree is the limit of the measures of its levels.

2-RAN, The Existence of 2-Randoms Formalized

Definition

2-RAN is RCA_0 augmented by the following assertion.

- For every set X there is a set R such that R passes every Martin-Löf test which is recursive in X'. *Caveat, X' may not exist as a set and the computation of the Martin-Löf test must converge uniformly.*

$C\Sigma_n$, the Cardinality Principle for Σ_n functions

Definition (Seetapun and Slaman, 1995)

$C\Sigma_n$ is the principle that asserts "If a Σ_n formula φ defines a function f from all the numbers into a bounded set, then f is not injective."

$C\Sigma_n$ sound bites:

- The infinite-to-finite pigeonhole principle for Σ_n partitions.
- The numbers form a Σ_n-cardinal.

The Cardinality Scheme and the Induction Hierarchy

Theorem (Conidis and Slaman, strengthening Kaye, 1997)

$$P^- + I\Sigma_n + \bigcup_{k\in\omega} C\Sigma_k \nvdash B\Sigma_{n+1}$$

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In particular, working over $P^- + I\Sigma_1$,

$$I\Sigma_1 \subset C\Sigma_2 \subset B\Sigma_2.$$

$P^- + I\Sigma_1 + \bigcup_{k\in\omega} C\Sigma_k \nvdash B\Sigma_2$

Proof.

Build a model $\mathfrak{N}_\omega$ as the direct limit of an ω-sequence $\mathfrak{N}_0 \prec_{\Sigma_n,e} \mathfrak{N}_1 \prec_{\Sigma_n,e} \ldots \mathfrak{N}_\omega$. Ensure the following.

- Each $\mathfrak{N}_i \models PA$ and has cardinality ω_i.
- $\mathbb{N}$ is Δ_2 in $\mathfrak{N}_\omega$.

□

An Instance of Non-Conservation

Theorem (Conidis and Slaman)

$\mathit{2}\text{-}RAN \vdash C\Sigma_2$

Proof.

Assume that there is a Σ_2-projection of all the numbers into the numbers less than a. Use the projection to define $0'$-recursive trees T_n of measure greater than $\frac{2^n - a}{2^n}$ such that for each σ of length n, T_n has only boundedly many extensions of σ. □

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A failure of $C\Sigma_2$ allows for a closed representation of a set of high measure by a finite union of empty sets.

Aside

$C\Sigma_2$ also follows from principles of other sorts such as the Rainbow Ramsey Theorem for 2-bounded colorings of pairs or the Thin Set Theorem for pairs.

A Conservation Theorem

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Proof.

It is sufficient to start with a countable model $\mathfrak{M} \models RCA_0 + B\Sigma_2$ in which there is a set A of greatest Turing degree and extend it to a model of $\mathit{2}\text{-}RAN + B\Sigma_2$.

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- There is a A'-recursive tree T_0 such that any unbounded path in T is 2-random over $\mathfrak{M}$.

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- Force with infinite A'-recursive subtrees of T_0 to add R.

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- There is a A'-recursive tree T_0 such that any unbounded path in T is 2-random over $\mathfrak{M}$.
- Force with infinite A'-recursive subtrees of T_0 to add R.
  - Verify $B\Sigma_1(R \oplus A')$ in $\mathfrak{M}[R]$.

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- There is a A'-recursive tree T_0 such that any unbounded path in T is 2-random over $\mathfrak{M}$.
- Force with infinite A'-recursive subtrees of T_0 to add R.
  - Verify $B\Sigma_1(R \oplus A')$ in $\mathfrak{M}[R]$.
- Recall, for reals R is 2-random(A), $(R \oplus A)' \equiv_T R \oplus A'$. Similarly, $B\Sigma_1(R \oplus A')$ is equivalent to $B\Sigma_2(R \oplus A)$ in $\mathfrak{M}[R]$.

□

2-RAN within the Paris-Kirby Hierarchy

We have the following information on the set of first order consequences of *2-RAN*.

- Lower bound, $I\Sigma_1 + C\Sigma_2$
- Upper bound, $B\Sigma_2$

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Now, we turn to an exact calculation.

The Pigeon Hole Principle for Measure

Definition

By the *Pigeon Hole Principle for Measure* we mean the principle that if $C \subseteq 2^\omega$ is a set of positive measure and $\ell \in \omega$, then there is a $\sigma \in 2^\ell$ such that $C \bigcap [\sigma]$ has positive measure.

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Definition

Let 2-$PHPM$ be the arithmetic principle that if T is a subtree of 2^ω which is recursive in $0'$ and of positive measure, then for every ℓ there is a $\sigma \in 2^\ell$ such that

$$T_\sigma = T \bigcap \{\tau : \sigma \text{ and } \tau \text{ are compatible}\}$$

has positive measure.

$\mathit{2\text{-}RAN}$ and 2-PHPM

Theorem

Suppose φ is an arithmetic sentence. Then

$$\mathit{2\text{-}RAN} \vdash \varphi \iff P^- + I\Sigma_1 + \mathit{2\text{-}PHPM} \vdash \varphi$$

$2\text{-}RAN$ and $2\text{-}PHPM$

Theorem

Suppose φ is an arithmetic sentence. Then

$$2\text{-}RAN \vdash \varphi \iff P^- + I\Sigma_1 + 2\text{-}PHPM \vdash \varphi$$

Proof.

For ($\Leftarrow$).

- For any countable model of $P^- + I\Sigma_1 + 2\text{-}PHPM$, there is a path R through the universal tree for 2-random reals.

□

$\mathit{2}$-RAN and 2-$PHPM$

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For ($\Leftarrow$).

- For any countable model of $P^- + I\Sigma_1 + 2\text{-}PHPM$, there is a path R through the universal tree for 2-random reals.
- Verify that 2-random sequences satisfy $I\Sigma_1$ and that their columns are mutually 2-random.

□

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Proof.

For $(\Leftarrow)$.

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- Verify that 2-random sequences satisfy $I\Sigma_1$ and that their columns are mutually 2-random.
- Conclude that the model with second order part the ideal of sets generated by the columns of R satisfies $\mathit{2\text{-}RAN}$.

□

The Strength of 2-$PHPM$

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- Build a model $\mathfrak{M}$ of 2-$PHPM + \neg B\Sigma_2$ as a direct limit of end-extensions of models of PA, reminiscent of the analysis of $\cup_{k\in\omega} C\Sigma_k$.

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- Build a model $\mathfrak{M}$ of 2-$PHPM + \neg B\Sigma_2$ as a direct limit of end-extensions of models of PA, reminiscent of the analysis of $\cup_{k\in\omega} C\Sigma_k$.
- For the directed system of end-extensions use a binary tree, which appears as a non-standard finite tree in an initial non-standard model of PA.

The Strength of 2-*PHPM*

Theorem

$P^- + I\Sigma_1 + \mathit{2}\text{-}PHPM \nvdash B\Sigma_2$.

- Build a model $\mathfrak{M}$ of 2-$PHPM$ $+ \neg B\Sigma_2$ as a direct limit of end-extensions of models of PA, reminiscent of the analysis of $\cup_{k \in \omega} C\Sigma_k$.
- For the directed system of end-extensions use a binary tree, which appears as a non-standard finite tree in an initial non-standard model of PA.
- Conclude that $\mathfrak{M}$ admits the existence of a 2-random predicate by applying the pigeon hole principle for measure in the intermediate models of PA which appear on the tree.

Summary

Working over RCA_0:

- Compactness arguments (WKL_0) have no number-theoretic strength.
- Category arguments (genericity) have no number-theoretic strength over $I\Sigma_1$ but can be used to compress the Kirby-Paris Hierarchy.
- Measure arguments (2-randomness) identify an fragment of arithmetic strictly between $I\Sigma_1$ and $I\Delta_2$ which is axiomitized by a first-order version of the pigeonhole principle for measure.

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