Automata on ordinals and linear orders

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Automata

We consider structures recognized by deterministic finite state automata.

Let us fix a finite alphabet $\mathcal{A}$. An automaton consists of

- a finite set S of states with specified initial state and accepting states and
- a transition function $\mathcal{A} \times S \to S$.

An input word is read from beginning to end and is accepted or rejected according to the final state.

Automata on ordinals

We consider automata with running time an arbitrary limit ordinal γ.

The extra ingredient is a limit transition function which maps the set of states appearing cofinally often (before a limit) to the limit state. A word is accepted or rejected according to the state at time γ.

Similar automata have been studied by Büchi, Choueka, Wojciechowski, and others.

Example

Suppose an ω^n-automaton moves into state 0 in successor steps and into state $m+1$ in limits, where m is the maximum of the states appearing cofinally often before the limit. This detects the limit type $m < n$ of the current step $\omega^{n-1} i_{n-1} + \omega^{n-2} i_{n-2} + \ldots + \omega^m i_m$ where $i_m \neq 0$.

Connection with monadic logic

A motivation for studying automata is the connection with monadic second order logic. For example, Büchi showed that every set defined by some monadic sentence over $(\omega, <)$ can be decided by an automaton.

Shelah proved that this connection breaks down at ω_2.

Neeman has found a much stronger type of automaton which corresponds to monadic sentences for all ordinals.

Automatic structures

Definition

An automatic structure is isomorphic to a structure $(M, R_0, ..., R_n)$ *such that* M *consists of finite words in a finite alphabet and* M *and* $R_0, ..., R_n$ *are recognized/accepted by finite state automata.*

We say that an ordinal δ is automatic if $(\delta, \leqslant)$ is automatic.

Let us fix a limit ordinal γ and an extra symbol $\diamond$.

A *finite* γ*-word* is a word of length γ such that only finitely many of its letters are not $\diamond$.

Definition

A γ*-automatic structure is isomorphic to a structure* $(M, R_0, ..., R_n)$ *such that* M *consists of finite* α*-words and* M *and* $R_0, ..., R_n$ *are recognized/accepted by* γ*-automata.*

Examples

Examples of automatic structures:

- $(\mathbb{N}, +, <)$
- $(\mathbb{Q}, <)$

It is usually harder to show that a structure is not automatic, for example:

- $(\mathbb{N}, \times)$
- $(\omega^\omega, <)$, the random graph (Delhommé)

Examples

Example of an ω^2-automatic structure:

Let $(n_0, ..., n_k) <^* (m_0, ..., m_l)$ if

- $k = l$ and $n_i < m_i$ for $i \leqslant n$ least with $n_i \neq m_i$, or
- $k < l$.

This is a wellorder on $\omega^{<\omega}$ of order type ω^ω. Represent $(n_0, ..., n_k)$ by $0^{n_0}1 \diamond^\omega 0^{n_1}1 \diamond^\omega ...0^{n_k}1\diamond^{\omega^2}$.

Similarly ω^β is α-automatic, where $\alpha = \omega \cdot \beta$.

Motivation

A motivation to study automatic structures: their theory is computable.

Sometimes easier classification:

- Automatic linear orders have finite Cantor-Bendixson rank (Khoussainov-Rubin-Stephan).
- The isomorphism problem for automatic ordinals is decidable (Khoussainov-Rubin-Stephan).
- The finitely generated automatic groups are exactly the finitely generated groups with an abelian subgroup of finite index (Oliver-Thomas).

However...

In some respects automatic structures can be complicated:

- There are automatic wellfounded relations of arbitrary large height below ω_1^{CK} and automatic structures with Scott rank $\omega_1^{CK} + 1$ (Khoussainov-Minnes).
- The isomorphism problem for automatic linear orders is Σ_1^1 complete (Kuske-Liu-Lohrey).

Ordinal automatic structures

Some of the properties persist for γ-automatic structures:

- The theory is computable.
- The class of γ-automatic structures is closed under finite products.
- Every $(\gamma \cdot n)$-automatic structure is γ-automatic.

The automatic ordinals are exactly those below ω^ω (Delhommé).
Which ordinals are γ-automatic?

Automatic ordinals

Proposition (Stephan-S.)

Suppose $\alpha = \omega \cdot \beta = \omega^\gamma$. *Then* $\omega^{\beta \cdot \omega}$ *is the supremum of the* α*-automatic ordinals.*

Hence the supremum of the ω^n-automatic ordinals is ω^{ω^n} and the supremum of the ω^α-automatic ordinals is $\omega^{\omega^{\alpha+1}}$ for $\alpha \geqslant \omega$.

The power of α-automata increases with every power of ω.

A finite type product

Suppose C, D are sets of ordinals. Let $tp(C)$ denote the order type of C. Let $tp(C, D)$ denote the isomorphism type of $(C \cup D, C, D, <)$.

Definition

$\alpha \otimes_{fin} \beta$ is the supremum (and maximum) of the ordinals γ such that there is a sequence $(C_\delta : \delta < \epsilon)$ with $\gamma = \bigcup_{\delta<\epsilon} C_\delta$ and

- $\forall \delta < \epsilon \ tp(C_\delta) \leqslant \alpha$,
- *there are only finitely many $tp(C_\delta, C_\eta)$ for $\delta, \eta < \epsilon$, and*
- *let for $\mu < \alpha$ $Tr_\mu = \{C_\delta(\mu) : \delta < \epsilon\}$ (the trace of μ). Then $\forall \mu < \alpha \ tp(Tr_\mu) \leqslant \beta$.*

A finite type product

This product is identical with the commutative product.

Definition

Suppose $\alpha = \Sigma_{i<m}\omega^{\alpha_i}$ *and* $\beta = \Sigma_{j<n}\omega^{\beta_j}$ *are in Cantor normal form, i.e.* $\alpha_0 \geqslant \ldots \geqslant \alpha_{m-1}$. *The commutative sum* $\alpha \oplus \beta$ *is the sum of all* ω^{α_i} *and* ω^{β_j} *arranged in decreasing order. The commutative product* $\alpha \otimes \beta$ *is the sum of all* $\omega^{\alpha_i \oplus \beta_j}$ *arranged in decreasing order.*

A finite type product

For $\alpha \otimes_{fin} \beta \leqslant \alpha \otimes \beta$ suppose $(C_\delta : \delta < \epsilon)$ is a sequence as in the definition of $\alpha \otimes_{fin} \beta$.

Case A: $tp(C_\gamma) = tp(C_\delta)$ and $\sup C_\gamma = \sup C_\delta$ for all $\gamma < \delta < \epsilon$. Then every proper inital segment of the product is bounded by $\bar{\alpha} \otimes_{fin} \beta = \bar{\alpha} \otimes \beta < \alpha \otimes \beta$ for some $\bar{\alpha} < \alpha = tp(C_\gamma)$.

Case B: $tp(C_\gamma) = tp(C_\delta)$ and $\sup C_\gamma < \min C_\delta$ or $\sup C_\delta < \min C_\gamma$ for all $\gamma < \delta < \epsilon$. Then every proper inital segment of the product is bounded by $\alpha \otimes_{fin} \bar{\beta} = \alpha \otimes \bar{\beta} < \alpha \otimes \beta$ for some $\bar{\beta} < \beta$.

Automatic ordinals

Proposition

Suppose $\alpha = \omega \cdot \beta = \omega^{\gamma}$. Then $\omega^{\beta \cdot \omega}$ is the supremum of the α-automatic ordinals.

Proof sketch:

Suppose there is an α-automatic wellorder of order type $\omega^{\beta \cdot \omega}$.

Pick u_n with $tp(u_n \downarrow) = \omega^{\beta \cdot n}$ for each $n \geqslant 1$, where $u_n \downarrow$ is the set of predecessors of u_n.

For each n

$$u_n \downarrow = X_{u_n} \sqcup \bigsqcup_{|v|=|u_n|} Y_v^{u_n}$$

where $X_{u_n} = \{x : |x| < |u_n| \ \& \ x < u_n\}$ and $Y_v^{u_n} = \{vw : vw < u_n\}$.

Automatic ordinals

$$u_n \downarrow = X_{u_n} \sqcup \bigsqcup_{|v|=|u_n|} Y_v^{u_n}$$

where $X_{u_n} = \{x : |x| < |u_n| \ \& \ x < u_n\}$ and $Y_v^{u_n} = \{vw : vw < u_n\}$.

Then $tp(X_{u_n}) < \omega^\beta$ since X_{u_n} is $|u_n|$-automatic and $|u_n| < \alpha$.

$\bigsqcup_{|v|=|u_n|} Y_v^{u_n}$ is a finite type product of the sets $Y_v^{u_n}$ and each $tp(Tr_\delta) < \omega^\beta$ since Tr_δ is $\bar{\alpha}$-automatic for some $\bar{\alpha} < \alpha$.

Hence there is v_n with $tp(Y_{v_n}^{u_n}) = \omega^{\beta \cdot n}$ for each n. But the number of $tp(Y_{v_n}^{u_n})$ is bounded by the number of states.

Linear orders

A linear order L is *scattered* it does not contain a copy of $(\mathbb{Q}, <)$.

Definition

Suppose L is a scattered linearly ordered set. Let rk$(L) = 0$ if L is finite. Let rk$(L) \leqslant \alpha$ if L is a $\mathbb{Z} \cdot n$-sum of linear orders of rank less than α for some n.

For an arbitrary linearly ordered set let $c(L)$ be the quotient of L where elements with finitely many elements in between them are identified. The rank of L is the least α such that $c^{\alpha}(L)$ does not have segments isomorphic to ω or ω^*.

Linear orders

Proposition (Stephan-S.)

Suppose $\alpha = \omega \cdot \beta = \omega^\gamma$. Then $\beta \cdot \omega$ is the supremum of ranks of α-automatic linear orders.

Definition

A linear order C is an finite type product of linear orders A and B if there are sequences $(C_\gamma : \gamma < \epsilon)$ of subsets of C and $(f_\gamma : \alpha \to C_\gamma : \gamma < \epsilon)$ of onto functions with

- $C = \bigcup_{\gamma<\epsilon} C_\gamma$
- $C_\gamma \hookrightarrow A$ *for all* $\gamma < \epsilon$
- *for each n, there are only finitely many* $tp(f_{\gamma_0}, ..., f_{\gamma_n})$ *for* $\gamma_i < \epsilon$.
- *let for* $\mu < \alpha$ $g_\mu(\gamma) = f_\gamma(\mu)$. *Then* $\forall \mu < \alpha$ $ran(g_\mu) \hookrightarrow B$.

Linear orders

Lemma

Suppose a scattered linear order C is a finite type product of A and B. Then $rk(C) \leqslant rk(A) \oplus rk(B)$.

Proposition

Suppose $\alpha = \omega \cdot \beta = \omega^\gamma$. *Then* $\beta \cdot \omega$ *is the supremum of ranks of* α*-automatic linear orders.*

Questions

What is the supremum of the ranks of γ-automatic wellfounded partial orders?

Is isomorphism of automatic scattered linear orders computable?