Reverse Mathematics & Nonstandard Analysis:

Making sense of infinite computations.

Sam Sanders[1]

Tohoku University & Ghent University

CiE, June 29, 2011, Sofia

[1]This research is generously supported by the John Templeton Foundation.

Goal & Motivation

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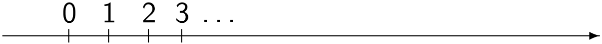

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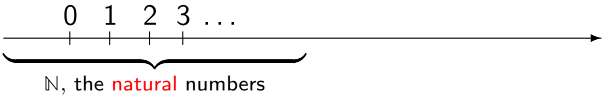

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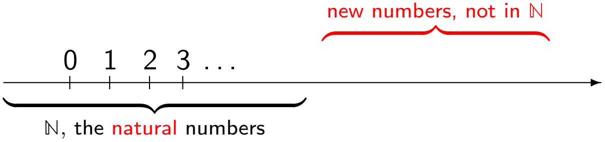

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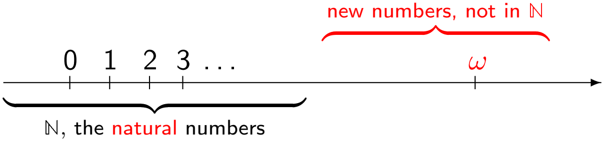

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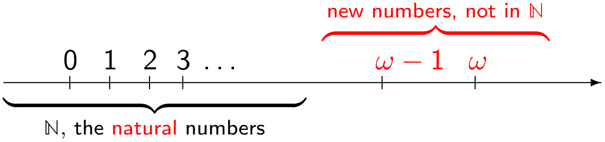

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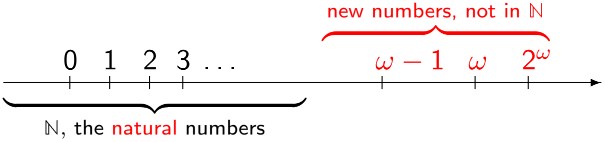

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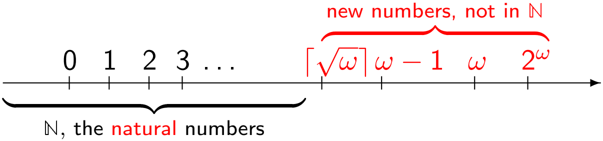

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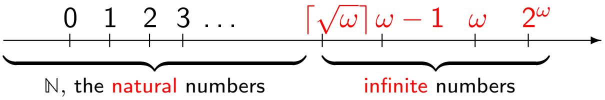

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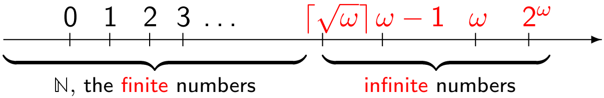

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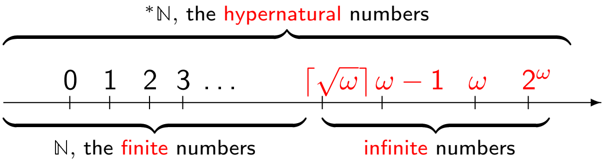

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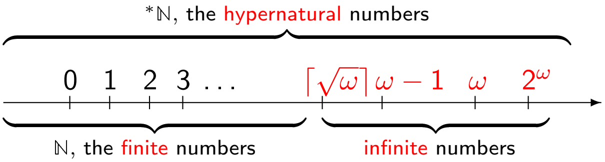

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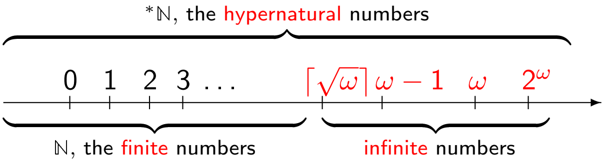

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Theorem

The Δ_1-sets (=Turing computable) are exactly the ω-invariant sets.

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For every $\varphi \in \Delta_0$, we have $(\forall n \in \mathbb{N})\varphi(n) \to (\forall n \in {}^*\mathbb{N})\varphi(n)$.

Also called 'Transfer principle for Π_1-formulas' or 'Π_1-transfer'.

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How about ERNA and Π_1-TRANS?

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An important principle is:

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For every q.f. formula φ, we have $(\exists n \in \mathbb{N})\varphi(n) \vee (\forall n \in \mathbb{N})\neg\varphi(n)$.

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LPO: $P \vee \sim P$, MP: $\sim\sim P \to P$, LLPO: $\sim(P \wedge Q) \to \sim P \vee \sim Q$ $(P, Q \in \Sigma_1)$

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Compare $\mathbb{N}$ and $\mathcal{N}$.

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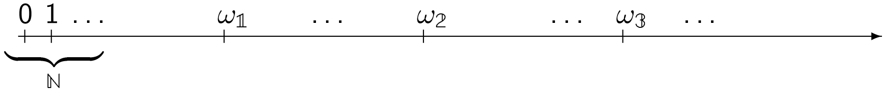

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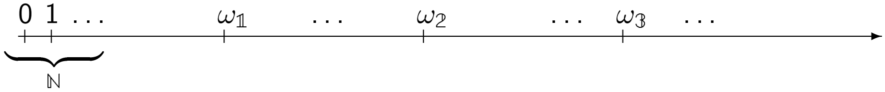

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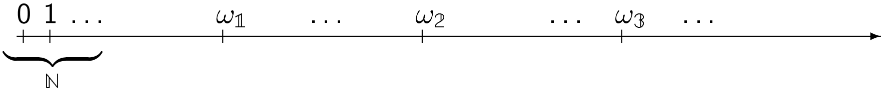

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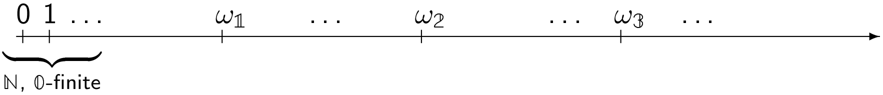

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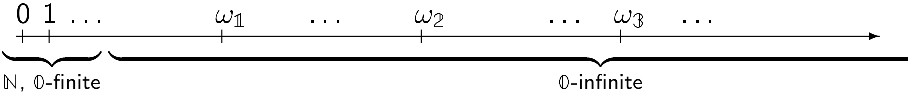

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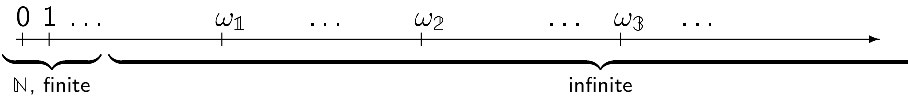

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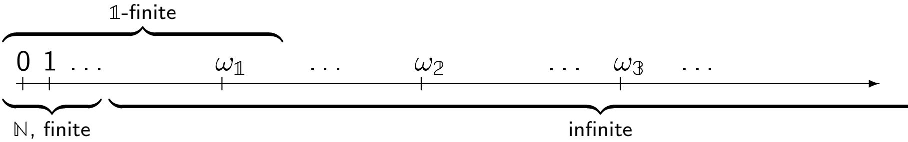

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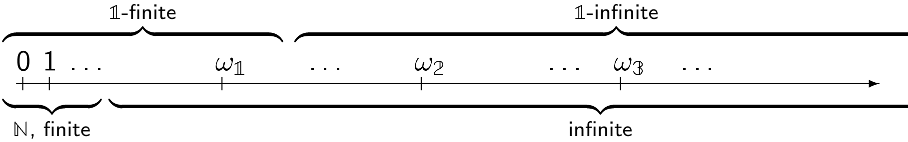

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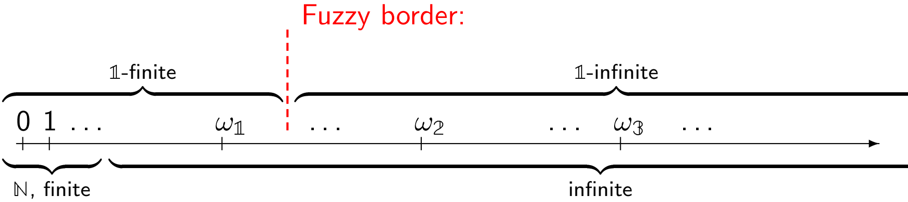

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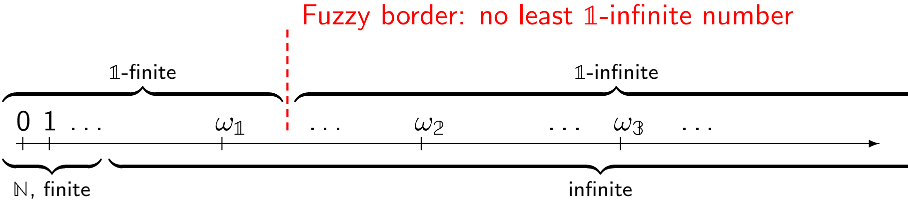

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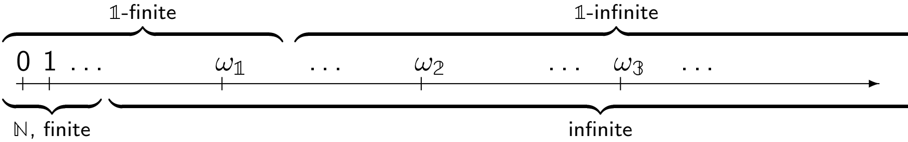

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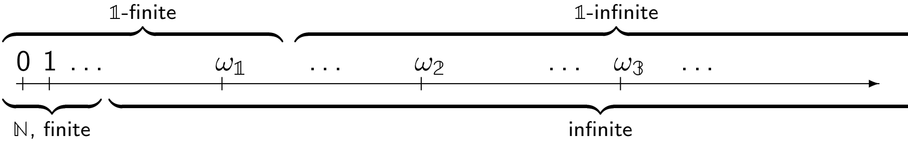

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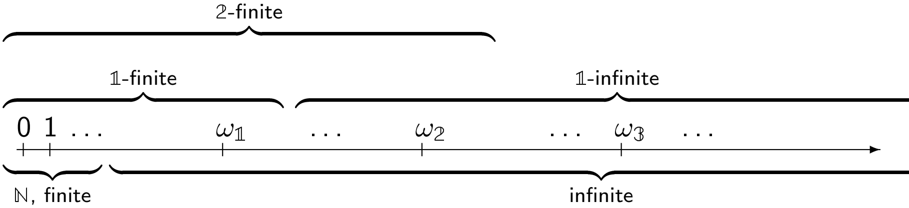

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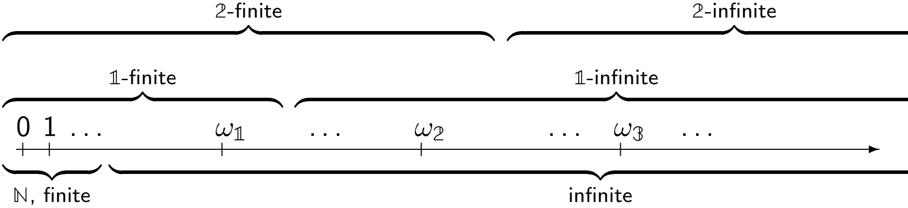

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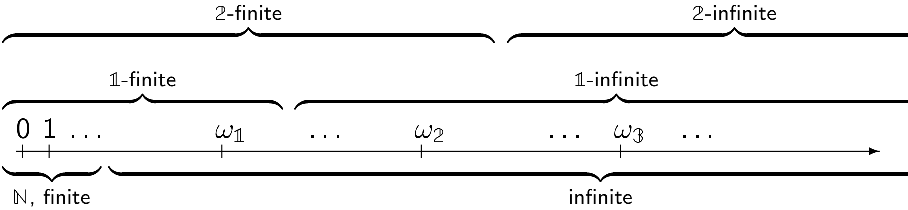

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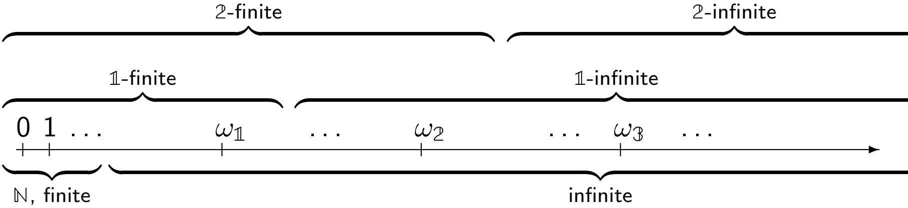

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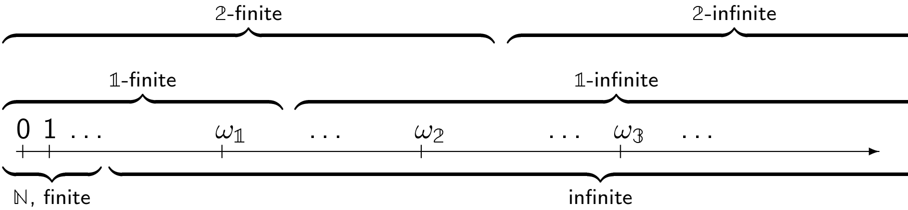

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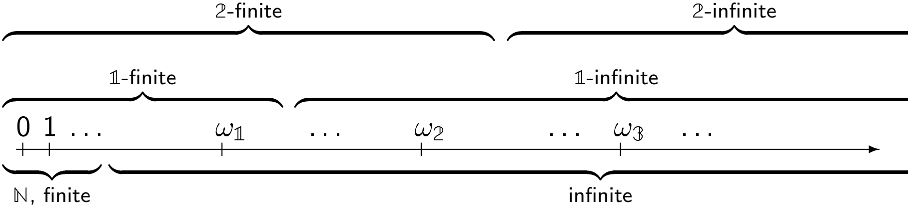

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