A quantitative nonlinear strong ergodic theorem for Hilbert spaces
or
An example of proof mining in ergodic theory

Pavol Safarik

Technische Universität Darmstadt, Germany

Computability in Europe 2011, Sofia

What more do we know if we have proved a theorem by restricted means than if we merely know that it is true.

Figure: Georg Kreisel

What more do we know if we have proved a theorem by restricted means than if we merely know that it is true.

Figure: Georg Kreisel

G. Kreisel was speaking of Unwinding proofs, the name proof mining was actually suggested by D. Scott.

The picture was taken during the Herbrand Centenary Lecture at Colloquium Logicum 2008 in Darmstadt.

1. Find a suitable theorem.
2. Analyze the proof.
3. Extract the computational content.

1. Find a suitable theorem.
2. Analyze the proof.
3. Extract the computational content.

1. Find a suitable theorem.
2. Analyze the proof.
3. Extract the computational content.

1. Find a suitable field in mathematics.
2. Find a suitable theorem.
3. Find a suitable proof.

1. Find a suitable theorem.
2. Analyze the proof.
3. Extract the computational content.

1. Find a suitable field in mathematics.
2. Find a suitable theorem.
3. Find a suitable proof.
4. Analyze the proof.
5. Extract the computational content.

1. Find a suitable theorem.
2. Analyze the proof.
3. Extract the computational content.



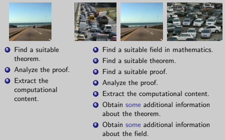
1. Find a suitable field in mathematics.
2. Find a suitable theorem.
3. Find a suitable proof.
4. Analyze the proof.
5. Extract the computational content.
6. Obtain some additional information about the theorem.
7. Obtain some additional information about the field.

1. Find a suitable theorem.
2. Analyze the proof.
3. Extract the computational content.



1. Find a suitable field in mathematics.
2. Find a suitable theorem.
3. Find a suitable proof.
4. Analyze the proof.
5. Extract the computational content.
6. Obtain some additional information about the theorem.
7. Obtain some additional information about the field.

In general one can almost always obtain computational information about the theorem, though ideally we hope to obtain uniformity results.



In general one never knows, though ideally we hope to obtain some sort of a logical pattern which assures the uniformity or the computability results.

1. Find a suitable theorem. *Logical Metatheorems*
2. Analyze the proof. *Proof Interpretations – Dialectica, Negative, n.c.i*
3. Extract the computational content. *Soundness of the Proof Interpretations*

Figure: Kurt Gödel

1. Find a suitable theorem. *Logical Metatheorems*
2. Analyze the proof. *Proof Interpretations – Dialectica, Negative, n.c.i*
3. Extract the computational content. *Soundness of the Proof Interpretations*

Figure: Kurt Gödel

Recall the talks of Ulrich Berger and Trifon Trifonov.

Recall the talk of Vasco Brattka - Computer Analysis in the Weihrauch Lattice, BW = Sigma 01 jump of WKL

Figure: J. von Neumann

Theorem (The Riesz version of the von Neumann mean ergodic theorem)

For any linear operator T on a Hilbert space X, which is nonexpansive, i.e.

$$\forall u, v \in X \ \left(\|Tu - Tv\| \leq \|u - v\|\right),$$

the sequence of the Cesàro means

$$A_n x := \frac{1}{n+1} \sum_{i=0}^{n} T^i x,$$

converges in norm for any starting point x.

It follows from an example by Genel and Lindenstrauss [Genel 1975] that there is a nonexpansive operator on the unit ball of ℓ_2, for which the sequence of the Cesàro means does not converge strongly.

Let H be a Hilbert space, C a subset of H and $T : C \to C$ a (possibly nonlinear) mapping.

Let H be a Hilbert space, C a subset of H and $T : C \to C$ a (possibly nonlinear) mapping.

Definition (Baillon 1975)

$$-C = C \text{ and } \forall u \in C \; \big(T(-u) = -Tu\big), \qquad \text{(odd)}$$

Let H be a Hilbert space, C a subset of H and $T : C \to C$ a (possibly nonlinear) mapping.

Definition (Baillon 1975)

$$-C = C \text{ and } \forall u \in C \ \big(T(-u) = -Tu\big), \tag{odd}$$

Definition (Brézis, Browder 1976)

$$\exists c \in \mathbb{R}\ \forall u, v \in C \tag{BB}$$
$$\big(\|Tu + Tv\|^2 \leq \|u + v\|^2 + c(\|u\|^2 - \|Tu\|^2 + \|v\|^2 - \|Tv\|^2)\big).$$

Let H be a Hilbert space, C a subset of H and $T : C \to C$ a (possibly nonlinear) mapping.

Definition (Baillon 1975)

$$-C = C \text{ and } \forall u \in C \ \big(T(-u) = -Tu\big), \tag{odd}$$

Definition (Brézis, Browder 1976)

$$\exists c \in \mathbb{R} \ \forall u, v \in C \tag{BB}$$
$$\big(\|Tu + Tv\|^2 \le \|u + v\|^2 + c(\|u\|^2 - \|Tu\|^2 + \|v\|^2 - \|Tv\|^2)\big).$$

Definition (Wittmann 1990)

$$\forall u, v \in C \ (\|T^n u + T^n v\| \le \alpha_n \|x + y\|), \tag{W^-}$$

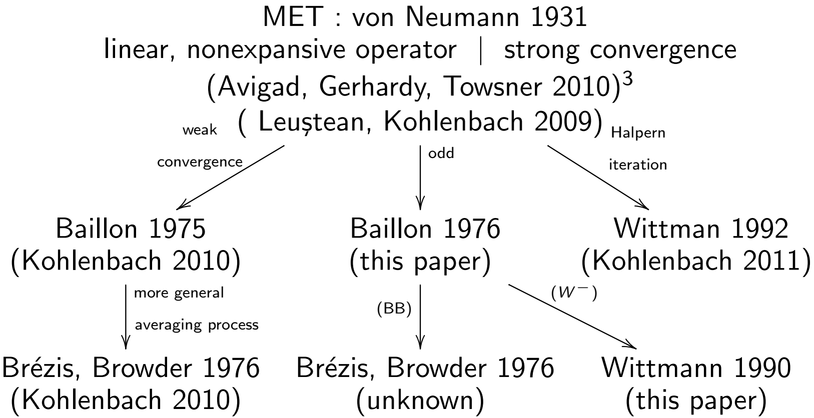


Figure: Nonlinear ergodic theorems and their finitisations.

Theorem (Wittmann 1990)

Let S be a subset of a Hilbert space and $T : S \to S$ be a mapping satisfying

$$\forall u, v \in C \; (\|T^n u + T^n v\| \leq \|x + y\|). \qquad \text{(W)}$$

Then for any $x \in S$ the sequence of the Cesàro means

$$A_n x := \frac{1}{n+1} \sum_{i=0}^{n} T^i x$$

is norm convergent.

In general, the sequence of the ergodic averages does not have a computable rate of convergence (even for the von Neumann's mean ergodic theorem for a separable space and computable x and T), as was shown by Avigad, Gerhardy and Towsner in 2008.

In general, the sequence of the ergodic averages does not have a computable rate of convergence (even for the von Neumann's mean ergodic theorem for a separable space and computable x and T), as was shown by Avigad, Gerhardy and Towsner in 2008.

This is the same reference as in the Figure above concerning the finitization of MET, see [1]

The metastable version has a primitive recursive bound.

The metastable version has a primitive recursive bound.

Theorem

Given the assumptions from Wittmann's strong ergodic theorem,

$$\forall u, v \in C \; (\|T^n u + T^n v\| \leq \|x + y\|). \tag{W}$$

the following holds

$$\forall b, l \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N}, x \in S \; \exists m \leq M(l, g, b) \\ \big(\|x\| \leq b \to \|A_m x - A_{m+g(m)} x\| \leq 2^{-l}\big),$$

for a primitive recursive M.

$$
\begin{aligned}
M(l, g, b) &:= (N(2l+7, g^M) + P(2l+7, g^M, b))b2^{2l+8} + 1, \\
P(l, g, b) &:= P_0(l, F(l, g, N(l, g), b), b), \\
F(l, g, n, b)(p) &:= p + n + \tilde{g}((n+p)b2^{l+1}), \\
N(l, g, b) &:= \big(H(l, g, b)\big)^{b^2 2^{l+2}}(0), \\
H(l, g, b)(n) &:= n + P_0(l, F(l, g, n, b)) + \tilde{g}((n + P_0(l, F(l, g, n, b)))b2^{l+1}),
\end{aligned}
$$

where

$$
P_0(l, f, b) := \tilde{f}^{b^2 2^l}(0), \quad \tilde{g}(n) := n + g(n), \; g^M(n) := \max_{i \le n+1} g(i).
$$

$$
\begin{aligned}
M(l,g,b) &:= (N(2l+7,g^M)+P(2l+7,g^M,b))b2^{2l+8}+1,\\
P(l,g,b) &:= P_0(l,F(l,g,N(l,g),b),b),\\
F(l,g,n,b)(p) &:= p+n+\tilde{g}((n+p)b2^{l+1}),\\
N(l,g,b) &:= \left(H(l,g,b)\right)^{b^2 2^{l+2}}(0),\\
H(l,g,b)(n) &:= n+P_0(l,F(l,g,n,b))+\tilde{g}((n+P_0(l,F(l,g,n,b)))b2^{l+1}),
\end{aligned}
$$

where

$$
P_0(l,f,b) := \tilde{f}^{b^2 2^l}(0), \quad \tilde{g}(n) := n+g(n),\ g^M(n) := \max_{i\le n+1} g(i).
$$

Note that apart from the counterfunction g and the precision l, this bound depends only on b and not on S, T or x.

The existence of such uniform bounds can be obtained by means of a general logical metatheorem.
(see [Kohlenbach 2005] and [Gerhardy, Kohlenbach 2008])

The existence of such uniform bounds can be obtained by means of a general logical metatheorem.
(see [Kohlenbach 2005] and [Gerhardy, Kohlenbach 2008])

Conditions to apply the metatheorems:

1. The proof does not use axioms or rules which are too strong.
2. The analyzed theorem in its logical form is not too complex in terms of quantification.

The existence of such uniform bounds can be obtained by means of a general logical metatheorem.
(see [Kohlenbach 2005] and [Gerhardy, Kohlenbach 2008])

Conditions to apply the metatheorems:

1. The proof does not use axioms or rules which are too strong.
2. The analyzed theorem in its logical form is not too complex in terms of quantification.

Theorem (Gerhardy-Kohlenbach 2008 - specific case 1)

Let $\varphi_\forall$, resp. $\psi_\exists$, be $\forall$- resp. $\exists$-formulas that contain only x, z, f free, resp. x, z, f, v free. Assume that $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle, S]$ proves the following sentence:

$$\forall x \in \mathbb{N}^{\mathbb{N}}, z \in S, f \in S^S \big(\varphi_\forall(x, z, f) \to \exists v \in \mathbb{N}\ \psi_\exists(x, z, f, v)\big).$$

Then there is a computable functional $F : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ s. t. the following holds in all non-trivial (real) inner product spaces $(X, \langle\cdot,\cdot\rangle)$ and for any subset $S \subseteq X$

$$\forall x \in \mathbb{N}^{\mathbb{N}}, z \in S, b \in \mathbb{N}, f \in S^S, f^* \in \mathbb{N}^{\mathbb{N}}$$
$$\big(\mathrm{Maj}(f^*, f)\ \wedge\ \|z\| \leq b\ \wedge\ \varphi_\forall(x, z, f) \to \exists v \leq F(x, b, f^*)\ \psi_\exists(x, z, f, v)\big),$$

where

$$\mathrm{Maj}(f^*, f) :\equiv \forall n \in \mathbb{N} \forall z \in S \big(\|z\| \leq_{\mathbb{R}} n \to \|f(z)\| \leq_{\mathbb{R}} f^*(n)\big).$$

Theorem (Gerhardy-Kohlenbach 2008 - specific case 1)

Let $\varphi_\forall$, resp. $\psi_\exists$, be $\forall$- resp. $\exists$-formulas that contain only x, z, f free, resp. x, z, f, v free. Assume that $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle, S]$ proves the following sentence:

$$\forall x \in \mathbb{N}^\mathbb{N}, z \in S, f \in S^S \big(\varphi_\forall(x, z, f) \to \exists v \in \mathbb{N}\ \psi_\exists(x, z, f, v)\big).$$

Then there is a computable functional $F : \mathbb{N}^\mathbb{N} \times \mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}$ s. t. the following holds in all non-trivial (real) inner product spaces $(X, \langle\cdot,\cdot\rangle)$ and for any subset $S \subseteq X$

$$\forall x \in \mathbb{N}^\mathbb{N}, z \in S, b \in \mathbb{N}, f \in S^S, f^* \in \mathbb{N}^\mathbb{N}$$
$$\big(\mathrm{Maj}(f^*, f) \ \wedge\ \|z\| \le b \ \wedge\ \varphi_\forall(x, z, f) \to \exists v \le F(x, b, f^*)\ \psi_\exists(x, z, f, v)\big),$$

where

$$\mathrm{Maj}(f^*, f) :\equiv \forall n \in \mathbb{N} \forall z \in S \big(\|z\| \le_\mathbb{R} n \to \|f(z)\| \le_\mathbb{R} f^*(n)\big).$$

The theorem holds analogously for finite tuples.

Wittmann's theorem has the following form:

$$\forall l \in \mathbb{N}, g \in \mathbb{N}^{\mathbb{N}}, x \in S, T \in S^S \qquad (+)$$
$$\big(\mathrm{W}(T) \to \exists m \in \mathbb{N} \; (\|A_m x - A_{m+g(m)} x\| < 2^{-l})\big).$$

Wittmann's theorem has the following form:

$$\forall l \in \mathbb{N}, g \in \mathbb{N}^{\mathbb{N}}, x \in S, T \in S^S \qquad (+)$$
$$\big(\mathrm{W}(T) \to \exists m \in \mathbb{N}\ (\|A_m x - A_{m+g(m)} x\| < 2^{-l})\big).$$

- the conclusion has the form $\exists m\ \psi_\exists(m, l, g)$
- the assumption $\forall x, y \in S\big(\|Tx + Ty\| \leq \|x + y\|\big)$ has the form $\varphi_\forall(T)$

Wittmann's theorem has the following form:

$$\forall l \in \mathbb{N}, g \in \mathbb{N}^{\mathbb{N}}, x \in S, T \in S^S \qquad (+)$$
$$\big(\mathrm{W}(T) \to \exists m \in \mathbb{N}\ (\|A_m x - A_{m+g(m)} x\| < 2^{-l})\big).$$

- the conclusion has the form $\exists m\ \psi_\exists(m, l, g)$
- the assumption $\forall x, y \in S\big(\|Tx + Ty\| \le \|x + y\|\big)$ has the form $\varphi_\forall(T)$

So by setting

$\underline{x} :=_{\mathbb{N}\times\mathbb{N}^{\mathbb{N}}} l, g,\ z :=_S x,\ f :=_{S\to S} T,\ f^* :=_{\mathbb{N}\to\mathbb{N}} \mathrm{id},\ \varphi_\forall(x, z, f) :\equiv \mathrm{W}(T),$

$$\exists v \in \mathbb{N}\ \psi_\exists(x, z, f, v) :\equiv \exists m \in \mathbb{N}\ \big(\|A_m x - A_{m+g(m)} x\| < 2^{-l}\big),$$

Wittmann's theorem has the following form:

$$\forall l \in \mathbb{N}, g \in \mathbb{N}^{\mathbb{N}}, x \in S, T \in S^S \qquad (+)$$
$$\big(\mathrm{W}(T) \to \exists m \in \mathbb{N}\ (\|A_m x - A_{m+g(m)}x\| < 2^{-l})\big).$$

- the conclusion has the form $\exists m\ \psi_\exists(m, l, g)$
- the assumption $\forall x, y \in S\big(\|Tx + Ty\| \le \|x + y\|\big)$ has the form $\varphi_\forall(T)$

So by setting

$\underline{x} :=_{\mathbb{N}\times\mathbb{N}^{\mathbb{N}}} l, g,\ z :=_S x,\ f :=_{S\to S} T,\ f^* :=_{\mathbb{N}\to\mathbb{N}} \mathrm{id},\ \varphi_\forall(x, z, f) :\equiv \mathrm{W}(T),$

$$\exists v \in \mathbb{N}\ \psi_\exists(x, z, f, v) :\equiv \exists m \in \mathbb{N}\ \big(\|A_m x - A_{m+g(m)}x\| < 2^{-l}\big),$$

we obtain that there is a computable bound $M : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$, s.t.

$$\forall l \in \mathbb{N}, g \in \mathbb{N}^{\mathbb{N}}, x \in S, T \in S^S$$
$$\big(\mathrm{W}(T) \wedge \|x\| \le b\ \to \exists m \le_{\mathbb{N}} M(l, g, b)\ (\|A_m x - A_{m+g(m)}x\| \le 2^{-l})\big).$$

Wittmann's theorem has the following form:

$$\forall l \in \mathbb{N}, g \in \mathbb{N}^{\mathbb{N}}, x \in S, T \in S^S \quad (+)$$
$$\big(\mathrm{W}(T) \to \exists m \in \mathbb{N}\ (\|A_m x - A_{m+g(m)}x\| < 2^{-l})\big).$$

- the conclusion has the form $\exists m\ \psi_\exists(m, l, g)$
- the assumption $\forall x, y \in S\big(\|Tx + Ty\| \leq \|x + y\|\big)$ has the form $\varphi_\forall(T)$

So by setting
$\underline{x} :=_{\mathbb{N}\times\mathbb{N}^{\mathbb{N}}} l, g,\ z :=_S x,\ f :=_{S\to S} T,\ f^* :=_{\mathbb{N}\to\mathbb{N}} \mathrm{id},\ \varphi_\forall(x, z, f) :\equiv \mathrm{W}(T),$

$$\exists v \in \mathbb{N}\ \psi_\exists(x, z, f, v) :\equiv \exists m \in \mathbb{N}\ \big(\|A_m x - A_{m+g(m)}x\| < 2^{-l}\big),$$

we obtain that there is a computable bound $M : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$, s.t.

$$\forall l \in \mathbb{N}, g \in \mathbb{N}^{\mathbb{N}}, x \in S, T \in S^S$$
$$\big(\mathrm{W}(T) \wedge \|x\| \leq b\ \to \exists m \leq_{\mathbb{N}} M(l, g, b)\ (\|A_m x - A_{m+g(m)}x\| \leq 2^{-l})\big).$$

$\mathrm{W}(T)$ already implies $\mathrm{Maj}(\mathrm{id}, T)$ (here id stands simply for the identity function on $\mathbb{N}$), since $\mathrm{W}(T)$ applied to $x = y = z$ implies

$$\forall z \in S\big(\|T(z)\| \leq \|z\|\big).$$

No.

No.

- the proof can be formalized in $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle, S]$

No.

- the proof can be formalized in $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle, S]$
- non-trivial principles needed in the proof are the existence of the infimum/supremum of bounded sequences and the principle of convergence for bounded monotone sequences.

No.

- the proof can be formalized in $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle, S]$
- non-trivial principles needed in the proof are the existence of the infimum/supremum of bounded sequences and the principle of convergence for bounded monotone sequences.
- Moreover, since the bound itself has only functions and numbers as arguments, it follows from Schwichtenberg 79 and Kohlenbach 99 that the bound is not only computable, but that the *bound is a primitive recursive functional in the sense of Gödel's $\mathcal{T}$*.

No.

- the proof can be formalized in $\mathcal{A}^\omega[X, \langle \cdot, \cdot \rangle, S]$
- non-trivial principles needed in the proof are the existence of the infimum/supremum of bounded sequences and the principle of convergence for bounded monotone sequences.
- Moreover, since the bound itself has only functions and numbers as arguments, it follows from Schwichtenberg 79 and Kohlenbach 99 that the bound is not only computable, but that the *bound is a primitive recursive functional in the sense of Gödel's T.*

Except for the question of the use of the axiom of extensionality (full extensionality is in general unavailable in any proof-theoretic extraction of computational bounds). Generally, one can avoid the use of full extensionality in proofs of statements about continuous objects. Note that in particular any nonexpansive operator is also continuous. However, in our case, the operator T may be discontinuous. Fortunately, Wittmann proves his main results as a consequence of a statement about a simple sequence of elements in S, which as such is independent of T (see Theorem 2.3 in [9]), whereby all relevant equalities are provable directly. Therefore the rule of extensionality suffices to formalize his proof.

No.

- the proof can be formalized in $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle, S]$
- non-trivial principles needed in the proof are the existence of the infimum/supremum of bounded sequences and the principle of convergence for bounded monotone sequences.
- Moreover, since the bound itself has only functions and numbers as arguments, it follows from Schwichtenberg 79 and Kohlenbach 99 that the bound is not only computable, but that the *bound is a primitive recursive functional in the sense of Gödel's $\mathcal{T}$.*

Hence the existence of a *uniform computable bound* for the metastable version can be inferred from the metatheorem in [2].

Furthermore, since the metatheorem is established by proof-theoretic reasoning, it provides not only the existence of a uniform bound but also a procedure for its extraction.

Now, in general such a bound might need so called bar-recursion (BR), which is required to interpret the schema of full comprehension over numbers in Spector's system (see [8]).

No.

- the proof can be formalized in $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle, S]$
- non-trivial principles needed in the proof are the existence of the infimum/supremum of bounded sequences and the principle of convergence for bounded monotone sequences.
- Moreover, since the bound itself has only functions and numbers as arguments, it follows from Schwichtenberg 79 and Kohlenbach 99 that the bound is not only computable, but that the *bound is a primitive recursive functional in the sense of Gödel's $\mathcal{T}$.*

Both of these principles need only bar-recursion restricted to numbers and functions ($BR_{0,1}$) and not full BR. (Kohlenbach shows in [6, 5] that both principles are provable from arithmetical comprehension which is interpreted in $\mathcal{T}_0 + BR_{0,1}$.)

The corresponding papers to Schwichtenberg 79 and Kohlenbach 99 are [7, 4]

The nonconstructive, or ineffective, content of Wittmann's proof are the principle of convergence for bounded monotone sequences of real numbers and the existence of infimum for bounded sequences of real numbers. For a given sequence, the ineffective principles can be replaced by weaker statements about natural numbers only.

The nonconstructive, or ineffective, content of Wittmann's proof are the principle of convergence for bounded monotone sequences of real numbers and the existence of infimum for bounded sequences of real numbers. For a given sequence, the ineffective principles can be replaced by weaker statements about natural numbers only.

In the presence of arithmetical comprehension, these weaker (arithmetical) statements are equivalent to the original (analytical) principles. For the convergence we work with the arithmetic Cauchy property and for infimum we give for any precision an approximate infimum.

The nonconstructive, or ineffective, content of Wittmann's proof are the principle of convergence for bounded monotone sequences of real numbers and the existence of infimum for bounded sequences of real numbers. For a given sequence, the ineffective principles can be replaced by weaker statements about natural numbers only.
In the presence of arithmetical comprehension, these weaker (arithmetical) statements are equivalent to the original (analytical) principles. For the convergence we work with the arithmetic Cauchy property and for infimum we give for any precision an approximate infimum.

1. Arithmetized convergence of a monotone bounded sequence $a_{(\cdot)}$:

$$\forall l \exists n \forall m \geq n \quad \left(|a_n - a_m| \leq 2^{-l}\right).$$

2. Arithmetized existence of the infimum of a bounded sequence $a_{(\cdot)}$:

$$\forall l \exists n \forall m \quad \left(a_n - a_m \leq 2^{-l}\right).$$

The nonconstructive, or ineffective, content of Wittmann's proof are the principle of convergence for bounded monotone sequences of real numbers and the existence of infimum for bounded sequences of real numbers. For a given sequence, the ineffective principles can be replaced by weaker statements about natural numbers only.
In the presence of arithmetical comprehension, these weaker (arithmetical) statements are equivalent to the original (analytical) principles. For the convergence we work with the arithmetic Cauchy property and for infimum we give for any precision an approximate infimum.

1. Arithmetized convergence of a monotone bounded sequence $a_{(\cdot)}$:
$$\forall l \exists n \forall m \geq n \quad \left(|a_n - a_m| \leq 2^{-l}\right).$$
2. Arithmetized existence of the infimum of a bounded sequence $a_{(\cdot)}$:
$$\forall l \exists n \forall m \quad \left(a_n - a_m \leq 2^{-l}\right).$$

(as opposed to statements about objects in $\mathbb{N}^{\mathbb{N}}$).

The nonconstructive, or ineffective, content of Wittmann's proof are the principle of convergence for bounded monotone sequences of real numbers and the existence of infimum for bounded sequences of real numbers. For a given sequence, the ineffective principles can be replaced by weaker statements about natural numbers only.
In the presence of arithmetical comprehension, these weaker (arithmetical) statements are equivalent to the original (analytical) principles. For the convergence we work with the arithmetic Cauchy property and for infimum we give for any precision an approximate infimum.

1. Arithmetized convergence of a monotone bounded sequence $a_{(\cdot)}$:
$$\forall l \exists n \forall m \geq n \quad (|a_n - a_m| \leq 2^{-l}).$$
2. Arithmetized existence of the infimum of a bounded sequence $a_{(\cdot)}$:
$$\forall l \exists n \forall m \quad (a_n - a_m \leq 2^{-l}).$$

While the analytical principles are actually known to be equivalent to arithmetic comprehension (see Simpson 99 and – for more detailed results – Kohlenbach 00), the arithmetic versions are equivalent to Σ^0_1-induction and hence have a functional interpretation by ordinarily primitive recursive functionals (see Kohlenbach 08).

The nonconstructive, or ineffective, content of Wittmann's proof are the principle of convergence for bounded monotone sequences of real numbers and the existence of infimum for bounded sequences of real numbers. For a given sequence, the ineffective principles can be replaced by weaker statements about natural numbers only.
In the presence of arithmetical comprehension, these weaker (arithmetical) statements are equivalent to the original (analytical) principles. For the convergence we work with the arithmetic Cauchy property and for infimum we give for any precision an approximate infimum.

1. Arithmetized convergence of a monotone bounded sequence $a_{(\cdot)}$:
$$\forall l \exists n \forall m \geq n \quad \left(|a_n - a_m| \leq 2^{-l}\right).$$
2. Arithmetized existence of the infimum of a bounded sequence $a_{(\cdot)}$:
$$\forall l \exists n \forall m \quad \left(a_n - a_m \leq 2^{-l}\right).$$

Formulated in the usual way, both principles state the existence of a real number, which we represent as fast converging Cauchy sequences of rationals[4] encoded as number theoretic functions (i.e. functions in $\mathbb{N}^{\mathbb{N}}$).

The nonconstructive, or ineffective, content of Wittmann's proof are the principle of convergence for bounded monotone sequences of real numbers and the existence of infimum for bounded sequences of real numbers. For a given sequence, the ineffective principles can be replaced by weaker statements about natural numbers only.
In the presence of arithmetical comprehension, these weaker (arithmetical) statements are equivalent to the original (analytical) principles. For the convergence we work with the arithmetic Cauchy property and for infimum we give for any precision an approximate infimum.

1. Arithmetized convergence of a monotone bounded sequence $a_{(\cdot)}$:
$$\forall l \exists n \forall m \geq n \quad (|a_n - a_m| \leq 2^{-l}).$$
2. Arithmetized existence of the infimum of a bounded sequence $a_{(\cdot)}$:
$$\forall l \exists n \forall m \quad (a_n - a_m \leq 2^{-l}).$$

Of course, in this way we don't get a single point which *is* the limit point or infimum. Therefore we have to analyze the proof and see whether such points are actually needed or whether these arithmetical versions suffice. Here, fortunately, it turns out that the latter is the case (see [3] for a general discussion of this point).

- Proof mining
- Ergodic theory
- Wittmann's theorem(s)
- Logical justification
- Some additional information about ergodic theory?

- Proof mining
- Ergodic theory
- Wittmann's theorem(s)
- Logical justification
- Some additional information about ergodic theory? maybe...

J. Avigad, P. Gerhardy, and H. Towsner.
Local Stability of Ergodic Averages.
Trans. Amer. Math. Soc., 362:261–288, 2010.

P. Gerhardy and U. Kohlenbach.
General logical metatheorems for functional analysis.
Trans. Amer. Math. Soc., 360:2615–2660, 2008.

U. Kohlenbach.
Elimination of Skolem functions for monotone formulas in analysis.
Arch. Math. Logic, 37(5-6):363–390, 1998.
Logic Colloquium '95 (Haifa).

U. Kohlenbach.
On the No-Counterexample Interpretation.
The Journal of Symbolic Logic, 64:1491–1511, 1999.

U. Kohlenbach.
Things that can and things that cannot be done in PRA.
Ann. Pure Applied Logic, 102:223–245, 2000.

U. Kohlenbach.
Applied Proof Theory: Proof Interpretations and their use in Mathematics.
Springer Monographs in Mathematics. Springer-Varlag, Berlin · Heidelberg · New York, 2008.

H. Schwichtenberg.
On Bar Recursion of Types 0 and 1.
The Journal of Symbolic Logic, 44:325–329, 1979.

C. Spector.
Provably recursive functionals of analysis: a consistency proof of an analysis by extension of principles formulated in current intuitionistic mathematics.
In *Recursive function Theory, Proceedings of Symposia in Pure Mathematics, vol. 5 (J.C.E. Dekker (ed.)), AMS, Providence, R.I.*, pages 1–27, 1962.

R. Wittmann.
Mean Ergodic Theorems for nonlinear operators.
Proceedings of the AMS, 108(3):781–788, 1990.