The rapid points of a complex oscillation

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Brownian motion

Definition 1. *Given a probability space* $(\Omega, \mathcal{B}, \mathbf{P})$*, a* Brownian motion *is a stochastic process* X *from* $\Omega \times [0,1]$ *to* $\mathbb{R}$ *satisfying the following properties:*

1. *Each path* $X(\omega, \cdot) : [0,1] \to \mathbb{R}$ *is almost surely continuous*
2. $X(\omega, 0) = 0$ *almost surely*
3. *For* $0 \leq t_1 < t_2 \cdots < t_n \leq 1$*, the random variables* $X(\omega, t_1), X(\omega, t_2) - X(\omega, t_1), \ldots, X(\omega, t_n) - X(\omega, t_{n-1})$ *are independent and normally distributed with mean* 0 *and variance* $t_1, t_2 - t_1, \ldots, t_n - t_{n-1}$*.*

Think of a hyperfinite random walk (normal random walks have to be carefully handled, since they generally do not converge to a Brownian motion if "squished"). Complex oscillations allow us to take certain events which happen *almost surely* for Brownian motion and ensure that they happen *definitely*.

Complex oscillations

For $n \geq 1$, we write C_n for the class of continuous functions on $[0,1]$ that vanish at 0 and are piecewise linear with slope $\pm\sqrt{n}$ on the intervals $[(i-1)/n, i/n]$, $i = 1, \ldots, n$. One can associate a binary string $a_1 \ldots a_n$ to every $x \in C_n$ by setting $a_i = 1$ or $a_i = -1$ according to whether x increases or decreases on the interval $[(i-1)/n, i/n]$. We call the word $a_1 \ldots a_n$ the *code* of x and denote it by $c(x)$. Conversely, every binary string s of length n clearly defines a unique element of C_n. We call a sequence (x_n) in $C[0,1]$ *complex* if $x_n \in C_n$ for each n and there is a constant d such that $K(c(x_n)) \geq n - d$ for all n. A function $x \in C[0,1]$ is a *complex oscillation* if there is a complex sequence (x_n) such that $\|x_n - x\|$ converges effectively to 0 (in the uniform norm) as $n \to \infty$.

Theorem 1. *[1] A continuous function on the unit interval is almost surely, with respect to Wiener measure, a complex oscillation.*

Importantly, Fouché [3] showed that there exists a recursive bijection between the set of KC-strings and an encoded version of the complex oscillations.

Hence, a complex oscillation can be considered as the uniform effective limit of a sequence of random walks encoded by a sequence of complex finite strings with a common compression coefficient.

Rapid points

Most points on the line are "ordinary" points of Brownian motion, as described by Khinchine's law of the iterated logarithm:

Theorem 2. *For X a one dimensional Brownian motion as above, we have that for any prescribed t_0 on* $[0, 1]$

$$\mathbf{P}\left\{\limsup_{h\to 0} \frac{X(t_0+h)-X(t_0)}{\sqrt{2|h|\log\log 1/|h|}} = 1\right\} = 1,$$

But, there are also uncountably many "rapid" points: t is called an α-rapid point of the sample path X if

$$\limsup_{h\to 0} \frac{|X(t+h)-X(t)|}{\sqrt{2|h|\log 1/|h|}} \geq \alpha.$$

The rapid points are those where the continuity of Brownian motion is pushed to the limit of its modulus of continuity. These turn out to have interesting dimensional properties.

Hausdorff dimension

Given a compact set A on the unit interval (or any bounded subset of $\mathbb{R}$) and $\epsilon > 0$, we consider all coverings of the set by open balls B_n of diameter smaller than or equal to ϵ. For each cover we form the sum

$$\sum_{n=0}^{\infty} |B_n|^\alpha,$$

where $|\cdot|$ denotes the diameter of a set (that is, the maximum distance between any two points). We will call these the α-Hausdorff sums for A, always with reference to a given cover. For each A we can take the infimum over all such sums, as $\{B_n\}$ ranges over all possible covers of A:

$$S_\alpha^\epsilon(B) = \inf_{\{B_n\}} \sum_n |B_n|^\alpha.$$

As ϵ decreases to 0, $S_\alpha^\epsilon(B)$ increases to a limit $H_\alpha(A)$ (which might be infinite) which is called the α-Hausdorff measure of A, or the Hausdorff measure of A in dimension α (we will refer to this as just "the measure" when the context is clear). Since H_α is σ-subadditive but otherwise satisfies the requirements of a measure, it is an outer measure.

Definition 2. *The* Hausdorff dimension*, $dimA$, of a compact set $A \subseteq [0,1]$ is the supremum of all the $\alpha \in [0,1]$ for which, for any cover B of A, $H_\alpha(B) = \infty$. This is equal to the infimum of all $\beta \in [0,1]$ for which there exists a cover C of A such that $H_\alpha(B) = 0$.*

We can think of assigning a "capacity" to each interval in a collection. We look at the infimum of the possibilities of the total number of intervals that still cover the set, and as we restrict the sizes of the intervals, this increases. The Hausdorff dimension is the point at which this total number falls from ∞ to 0.

The α-rapid points of a Brownian motion have a Hausdorff (and Fourier) dimension of $1 - \alpha^2$, almost surely. Fouché [3] showed that the law of the iterated logarithm holds for complex oscillations at all recursive numbers, and Kjos-Hanssen and Nerode [5] showed that it holds almost surely on all numbers in $[0,1]$. So do the rapid points have the same Hausdorff dimension as Brownian motion?

Tools: Effective generating sequences

If F is a subset of $C[0,1]$, we denote by $\overline{F}$ the topological closure of F in $C[0,1]$. For $\varepsilon > 0$ we let $O_\varepsilon(F)$ be the set $\{f \in C[0,1] : \exists_{f \in F} \|f - g\| < \varepsilon\}$. In the sequel, the complement of F is denoted by F^0 and F by F^1.

Definition 3. *(Fouché [3]) A sequence $\mathcal{F}_0 = (F_i : i < \omega)$ in Σ (the Borel subsets of $C[0,1]$) is an effective generating sequence if*

1. *for $F \in \mathcal{F}_0, \varepsilon > 0$ and $\delta \in \{0,1\}$, we have, for $G = O_\varepsilon(F^\delta)$ or $G = F^\delta$, that $W(\overline{G}) = W(G)$;*
2. *there is an effective procedure that yields, for each sequence $0 \leq i_1 < \cdots < i_n < \omega$ and $k < \omega$, a binary rational number β_k such that*

$$|W(F_{i_1} \cap \cdots \cap F_{i_n}) - \beta_k| < 2^{-k};$$

3. *for $n, i < \omega$, a strictly positive rational number ε and $x \in C_n$, both the relations $x \in O_\varepsilon(F_i)$ and $x \in O_\varepsilon(F_i^0)$ are recursive in x, ε, i and n.*

Simply put, we can consider an EGS as a sequence of sets having the following properties:

1. The sets and their complements (and their ε-neighbourhoods) have the same Wiener measure as their closures, which implies they do not have strange boundaries.

2. We can effectively determine the measure of finite intersections of sets.

3. Given a set in the EGS and a random walk (of the type used to construct complex oscillations), we can effectively determine whether they the random walk is in the set or a rationally determined neighbourhood, and the same for the complement of the set.

Let $\mathcal{F}$ be the effectively generated algebra generated by $\mathcal{F}_0$. For a total recursive function $\phi : \omega \to \omega$ and some effective enumeration (T_i) of $\mathcal{F}$, we say the sequence $(T_{\phi(n)})$ is $\mathcal{F}$*-semi-recursive*. The union of an $\mathcal{F}$-semi-recursive sequence over all n is termed a $\sum_1^0(\mathcal{F})$ set. If for a sequence (B_n) of sets in $\mathcal{F}$ there exists a total recursive function $\phi : \omega^2 \to \omega$ and an effective enumeration (T_i) of $\mathcal{F}$ such that each B_n can be described as $\bigcup_m T_{\phi(n,m)}$, it is called a uniform sequence of $\sum_1^0(\mathcal{F})$ sets.

Theorem 3. *[2] If (A_k) is a uniform sequence of $\Sigma_1^0(\mathcal{F})$ sets with $\sum_k W(A_k) < \infty$, then, for each complex oscillation x, it is the case that $x \notin A_k$ for all large values of k.*

The rapid points of complex oscillations

Theorem 4. *The α-rapid points of any complex oscillation have Hausdorff dimension $1-\alpha^2$.*

We describe the method of proof. Using a formulation of Hausdorff dimension from nonstandard analysis, we can show that counting certain intervals is sufficient to obtain accurate estimates of Hausdorff dimension. The main problem is then to show that the intervals we need can be described in a suitable way using effective generating sequences.

Fouché [2] has described how sets of the form $[\sup\{X(t) : t \in I\} \geq b]$ can be used to form an EGS (where X is a Brownian motion, I is an interval with recursive endpoints and b is a recursive number). We form, for each interval $[i2^{-k}, (i+1)2^{-k}]$, the new Brownian motion $Y_{k,i}(t) = X(t) - X(i2^{-k})$. Our EGS is then composed of events of the form

$$[\sup\{Y_{k,i}(t) : t \in [(i+1)2^{-k} - 2^{-k-j}, (i+1)2^{-k}]\} \geq \beta 2^{-\frac{k}{2}} \sqrt{2k \log 2}]$$

for some fixed rational $\beta < \alpha$. Once we have this, we can construct the event of having at least $2^{(1-\alpha^2-\varepsilon)n}$ "rapid intervals", that is, intervals which have a large fluctuation between their start and end, out of the 2^n possible intervals (for arbitrarily large n).

It can be shown that the event of this *not* happening can be described as a uniform sequence of $\Sigma_1^0(\mathcal{F})$ sets which form a converging sum (using approximations for Brownian motion). It follows that each complex oscillation drops out of the sequence at some time, implying we will have many rapid intervals for arbitrarily large n. This implies the required Hausdorff dimension of at least $1 - \alpha^2$.

The proof that the Hausdorff dimension is at most $1 - \alpha^2$ proceeds analogously.

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