Unbounded effectiveness: Turing's departure from Hilbert's desiderata

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- thanks to the useful comments from the reviewers which lead us in a different direction.

Outline

1. Historical background

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- This view is partially supported by the success of Turing's solution
- But the story is far from simple as many have shown
- While many accounts emphasize what was recovered in Turing's definition, we want to talk about what he left out.

Historical background

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- Other less well known names are Thue, Kronecker and Pasch, but nonetheless they have received some attention

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- In its original formulations, it asks for an effective procedure to decide the validity of first order formulae.
- But of course, it requires an implicit definition of an effective procedure that would satisfy the requirements of Hilbert's program.

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4. it should be possible to set an upper bound on the number of steps needed.

But it is unclear if he really wanted it in the list. In any case, he didn't mentioned it again.

Pasch pre-dated Hilbert

- But Hilbert was not alone
- In the XIXth century, Thue made a similar point, but regarding solutions of a certain type of combinatorial problems.
- According to Pasch, Kronecker did the same, but in a more general way.
- And Pasch himself did it, around 1904-1907, while preparing his *Foundations of Analysis*.


Kronecker

Pasch's statement of the requirement I

"As is well known, it was Kronecker (1823–1891) who first proposed the decidability requirement and argued that any concept whose definition is not supported by a proof of decidability is to be discarded. Each definition then, is to include a procedure that, in each case where the definition is applicable, yields a series of inferences the last of which determines whether the given case satisfies the definition."

Pasch's statement of the requirement II

"The procedure will be useful only if it yields, in each case, a finite series of inferences; i.e., only if the general procedure always allows us to calculate an upper bound for the number of inferences needed in a particular case."

Is it a requirement for acceptable mathematics?

"This convinced me that I had only two choices: either accept the requirement or distinguish clearly between the areas that satisfy it and those that do not. We might then distinguish between settled and unsettled areas. In my discussion of the decidability question in *Variable and Function*, I make a distinction between calculation satisfying the decidability requirement and 'improper calculation.'"

Hilbert's formulation

Referring to an instance of an effective procedure, Hilbert said:

"Instead, new principles and considerations of a completely different sort were necessary in order to show that the constructions of the full system of invariants requires only a finite number of operations, and that this number is less than a bound that can be stated *before* the calculation."

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which sounds pretty close to what computational (time)-complexity theory is about, admittedly in a very naive way.

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- and (more controversially) that he met requirement (3) only partially...
- and he proceeded this way so that a machine with an infinite tape could accommodate finite but unrestricted calculations...
- but in Turing's paper there is no attempt to deal with the fourth requirement.

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- but it is not implausible that Pasch (and Kronecker?) would be willing to pay the price to be in the "settled areas of mathematics"
- and we ourselves do pay the price when we talk about how computational classes tell the feasible from the infeasible.

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- Of course, in the absence of any evidence (and we are not sure about it), we can only speculate on why Turing did not take (4) into account
- But we can imagine otherwise, and muse about an alternative history of computability and complexity theory
- In this history, Turing could have considered topics akin to those of computational complexity decades before the actual birth of the subject
- However, this hypothetical path could also have complicated (or at least delayed) his definition in an unpredictable way, as he would have had to surmount enormous technical obstacles to succeed

Conclusions and further work

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- And a obvious future line of research is to clarify all the fine historical details of our paper …

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