The jump of a structure.

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Basic definitions in computability

For $A, B \subseteq \mathbb{N}$, A is *B-computable* ($A \leq_T B$) if
there is a computable procedure that answers "$n \in A$?",
using B as an *oracle*.

We impose no restriction on time or space.

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Obs: A is B-computable $\iff$ A and $(\mathbb{N} \setminus A)$ are both B-c.e.

The join

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Via an effective bijection $\mathbb{N} \leftrightarrow \mathbb{N}^2$, we view $\bigoplus_n A_n$ as $\subseteq \mathbb{N}$.

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We define the *jump* of a set A:

$A' = \{\ulcorner p \urcorner : p$ is a program that halts with oracle $A\}$

$\equiv_T \{\ulcorner \varphi \urcorner : \varphi$ is a quantifier-free formula s.t. $(\mathbb{N}, A) \models \exists x \varphi(x)\}$

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Thm: (Jump inversion theorem, [Friedberg 57])

If $A \geq_T 0'$, then there exists B such that $B' \equiv_T A$.

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Various areas have been studied,

1. Combinatorics,
2. Algebra,
3. Analysis,
4. Model Theory

In many cases one needs to develop a better understanding of the mathematical structures to be able to get the computable analysis.

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Note: A countable $\mathbb{Q}$-vector space $\mathcal{V} = (V, 0, +_v, \cdot_v)$ can be encoded by three sets: $V \subseteq \mathbb{N}$, $+_v \subseteq \mathbb{N}^3$ and $\cdot_v \subseteq \mathbb{Q} \times \mathbb{N}^2$.

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Theorem:
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However, basis can be found computable in $\mathbf{0}'$.
Moreover, $\exists$ comp. vector sp., all whose basis compute $\mathbf{0}'$.

Representing Structures

Def: By *structure* we mean a tuple $\mathcal{A} = (A; P_0, P_1, ..., f_0, f_1, ..)$
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Def: The *spectrum* of the isomorphism type of $\mathcal{A}$:
$Sp(\mathcal{A}) = \{X \subseteq \mathbb{N} : X \text{ computes a copy of } \mathcal{A}\}$.

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Let $\mathcal{A}$ be a structure.

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R.I.C.E. – a frequently re-discovered concept

Thm: [Ash, Knight, Manasse, Slaman; Chishholm][Vaĭtsenavichyus][Gordon]
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r.i.c.e. relations on $\mathcal{A}$ are the analog of c.e. subsets of $\mathbb{N}$.

We now want a *complete* r.i.c.e. relation.

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Example: In particular $\overrightarrow{0'}$ is r.i.c.e. in $\mathcal{A}$.

The upper-semi lattice of sequences of relations – à la Soskov's structure-degrees

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Obs: $X \leq_T Y \Longrightarrow \vec{X} \leq_s^{\mathcal{A}} \vec{Y}$.

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Obs: $\vec{K}^{\mathcal{A}}$ is *complete among r.i.c.e. sequences* of relations in $\mathcal{A}$.
I.e. If $\vec{Q}$ is r.i.c.e., there is $\bar{a} \in A^{<\omega}$ and a computable $f : \mathbb{N} \to \mathbb{N}$ s.t.

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I.e. If $\vec{Q}$ is r.i.c.e., there is $\bar{a} \in A^{<\omega}$ and a computable $f\colon \mathbb{N} \to \mathbb{N}$ s.t.
$\forall \bar{b} \forall i\ (\bar{b} \in Q_i \iff (\bar{a}, \bar{b}) \in K_{f(i)})$

Definition

Given $\vec{Q}$, let $\vec{Q}'^{\mathcal{A}}$ be $\vec{K}^{(\mathcal{A},\vec{Q})}$.

Note: $\vec{K}^{\mathcal{A}} = \emptyset'^{\mathcal{A}}$.
Note: We can also define $\vec{Q}''^{\mathcal{A}}$ as $\vec{K}^{(\mathcal{A},\vec{Q}'^{\mathcal{A}})}$.

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Ex: Let $\mathcal{A} = (A, \equiv)$ where $\equiv$ is an *equivalence relation*. Then
$\emptyset'^{\mathcal{A}} \equiv_s^{\mathcal{A}} (E_k(x) : k \in \mathbb{N}) \oplus \vec{R} \oplus \overrightarrow{0'}$,
where $E_k(x) \iff$ there are $\geq k$ elements equivalent to x,
and $R = \{\langle n, k\rangle \in \mathbb{N}^2 :$ there are $\geq n$ equivalence classes with $\geq k$ elements$\}$.

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There were various independent definitions of the jump of a structure $\mathcal{A}'$:

- Baleva.
    - domain: Moschovakis extension of $\mathcal{A} \times \mathbb{N}$.
    - relation: add a universal computably infinitary Σ_1 relation.
- I. Soskov.
    - domain: Moschovakis extension of $\mathcal{A}$.
    - relation: add a predicate for forcing Π_1 formulas.
- Stukachev. considered arbitrary cardinality, and Σ-reducibility
    - domain: Hereditarily finite extension of $\mathcal{A}$, $\mathbb{HF}(\mathcal{A})$.
    - relation: add a universal finitary Σ_1 relation.
- Montalbán. The definition above.

Computational-reductions between structures

Let $\mathcal{A}$ and $\mathcal{B}$ be structures.
Recall: $Sp(\mathcal{A}) = \{X \subseteq \mathbb{N} : X \text{ computes a copy of } \mathcal{A}\}$.

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Obs: $\mathcal{A} \leq_I \mathcal{B} \implies \mathcal{A} \leq_\Sigma B \implies \mathcal{A} \leq_w \mathcal{B}$.

Three main theorems about the jump

- 1st Jump inversion theorem.
- 2nd Jump inversion theorem.
- Fixed point theorem.

First Jump Inversion Theorem

Theorem (1st Jump inversion Theorem)

If $\overrightarrow{0'}$ is r.i.computable in $\mathcal{A}$,
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Question:
Which structures are $\equiv_l$-equivalent to the jump of a structure?

First Jump Inversion Theorem – applications

Theorem (1st Jump inversion Theorem - α-iteration)

If $\overrightarrow{0^{(\alpha)}}$ is r.i.computable in $\mathcal{A}$,
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[Greenberg, M, Slaman] used to build a structure whose spectrum is non-HYP

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It follows that r.i.Σ^0_n relations are Σ^c_n-definable.

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Cor:[M] Given $\mathcal{A}$, the following are equivalent:

- Low property: If $X \in Sp(\mathcal{A})$ and $X' \equiv_T Y'$ then $Y \in Sp(\mathcal{A})$.
- Strong jump inversion: If $X' \in Sp(\mathcal{A}')$ then $X \in Sp(\mathcal{A})$.

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Idea of proof: Build $\mathcal{A}$ as a non-well-founded ω-model of $V = L$ such that for some $\alpha \in \mathcal{A}$, $\mathcal{A} \cong L_\alpha^{\mathcal{A}}$.

Complete sets of Σ_n^c relations

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Definition (M.)

$P_0, ..., P_k, ...$ are a *complete set of Σ^c_n relations on $\mathcal{A}$* if they are uniformly Σ^c_n and $\bigoplus_k P_k \oplus \overrightarrow{0^{(n)}} \equiv^{\mathcal{A}}_s \emptyset^{(n)\mathcal{A}}$.

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Theorem (K.Harris – M. 08)

On Boolean algebras, $\forall n \in \mathbb{N}$, there is a finite sequence $P_0, .., P_{k_n}$, s.t. for all $\mathcal{A}$

$$\emptyset^{(n)^{\mathcal{A}}} \equiv_s^{\mathcal{A}} P_0(x) \oplus ... \oplus P_{k_n}(x) \oplus \overrightarrow{0^{(n)}}.$$

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Thm: [M.] There is no relativizable (and hence nice) set of Σ^c_3 relations that work for all linear orderings simultaneously.

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$\emptyset'^{\mathcal{A}} \equiv^{\mathcal{A}}_{s} \vec{LD} \oplus \overrightarrow{0'}$

where $\vec{LD} = (LD_1, LD_2, ...)$, and $LD_i = \{(v_1, ..., v_i) : v_1, ..., v_i$ are lin. dep.$\}$

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$$\emptyset'^{\mathcal{A}} \equiv_s^{\mathcal{A}} \vec{LD} \oplus \overrightarrow{0'}$$

where $\vec{LD} = (LD_1, LD_2, ...)$, and $LD_i = \{(v_1, ..., v_i) : v_1, ..., v_i \text{ are lin. dep.}\}$

Thm:[Knight-R. Miller-M.-Soskov-Soskova-Soskova-VanDendreissche-Vatev]
No finite set of relations is Σ_1^c complete in $\mathcal{V}$.

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Ex: $\emptyset'^{\mathcal{A}} \equiv^{\mathcal{A}}_{s} (E_k(x) : k \in \mathbb{N}) \oplus \overrightarrow{R} \oplus \overrightarrow{0'}$,
where $E_k(x) \iff$ there are $\geq k$ elements equivalent to x,
and $R = \{\langle n, k\rangle \in \mathbb{N}^2$: there are $\geq n$ equivalence classes with $\geq k$ elements$\}$.

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Suppose that $\mathcal{A}$ has infinitely many classes of each size.

Thm:[Knight-R. Miller-M.-Soskov-Soskova-Soskova-VanDendreissche-Vatev]

No finite set of relations is Σ^c_1 complete in $\mathcal{A}$.

Nice description of jump VS coding information.

Theorem ([M])

Let $\mathbb{K}$ be an axiomatizable class of structures.
Exactly one of the following holds:

(relative to any sufficiently large oracle)

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2. *Every set can be coded in $\mathcal{A}^{(n-1)}$:*

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2. *Every set can be coded in $\mathcal{A}^{(n-1)}$:*
   - *$\forall X \subseteq \omega$, there is a $\mathcal{A} \in \mathbb{K}$ s.t. X is a r.i.c.e. real in $\mathcal{A}^{(n-1)}$,*
   - *There is no uniform, rel, countable complete sets of Σ^c_n rels.*
   - *$\exists$ Continuum many n-back-and-forth equivalence classes*