preamble

Characterizing randomness by integral tests

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Main thesis

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Randomness

is equivalent to

Differentiability

by Demuth, Pathak, Nies et al.

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Then why?

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Then why?
The Lebesgue Differentiation Theorem is (a part of) an explanation.

My suggestion

A test

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Integration

is closed related with

Differentiation.

Differentiability

More precisely ...

A way from randomness to differentiability

Differentiability

More precisely ...
A way from randomness to differentiability

- pass a test,
- finite for an integral test,
- LDT holds for an integral test,
- differentiable for f s.t. f' is an integral test,
- their "difference" versions.

Integral tests

randomness notion	integral test
weak 2-randomness	a.e. finite
Martin-Löf	c.e.
computably	?
Schnorr	c.e. with a comp. integration
Kurtz	computable

Contents

- algorithmic randomness notions

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- extended computable functions from $[0, 1]$ to $[0, +\infty]$

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- characterizations in terms of integral tests

Martin-Löf randomness

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μ: uniform (Lebesgue) measure
A *c.e.* open set is a union of a c.e. set of cylinders.

Definition 1. *A* Martin-Löf test *(or ML-test) is a sequence* $\{U_n\}$ *of uniformly c.e. open sets with* $\mu(U_n) \leq 2^{-n}$.
A *passes a ML-test if* $A \notin \bigcap_n U_n$.
A Martin-Löf random *if it passes all Martin-Löf tests.*

Integral test

We identify 2^ω with $[0, 1]$.

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Definition 2. $t : [0, 1] \to [0, +\infty]$ *is c.e. if*

$$\{x \ : \ q < t(x)\}$$

are c.e. open uniformly in $q \in \mathbb{Q}$*.*

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Proposition 4. *TFAE:*

- *A real* z *is ML-random.*
- $t(z) < \infty$ *for all integral tests.*

Differentiability

f is of *bounded variation* if

$$\sup \sum_{i=1}^{n} |f(t_{i+1}) - f(t_i)| < \infty$$

where $t_1 < t_2 < \ldots < t_n$ in $[0, 1]$.

Theorem 5 (Demuth; Nies, Brattka and Miller). $z \in [0, 1]$ *is ML-random* $\iff$ *every comp. func.* f *of bounded variation is differentiable at* z.

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Relation btw. integral tests and differentiability?

Versions of integral tests

Weak 2-randomness

Definition 6. generalized ML-test*: $\{U_n\}$ of uniformly c.e. open sets with $\lim_n \mu(U_n) = 0$.*
z *is* weakly 2-random *if it passes all generalized ML-tests.*

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z is weakly 2-random *if it passes all generalized ML-tests.*

Theorem 7. *TFAE:*

- *A real z is weakly 2-random.*
- *$t(z) < \infty$ for all c.e. functions t such that $t(x) < \infty$ a.e.*
- *$t(z) < \infty$ for all c.e. functions t such that $\int f \circ t(x)dx < \infty$ for some order function f.*

Proof

Theorem 8. *z is weakly 2-random iff $t(z) < \infty$ for all c.e. functions t such that $t(x) < \infty$ a.e.*

Proof. $\{U_n\}$: a decreasing generalized ML-test
Let $t(x) = \sup_n\{n \ : \ x \in U_n\}$.
Let $U_n = \{x \ : \ t(x) > n\}$. □

Schnorr randomness

Definition 9. Schnorr test*: a ML-test s.t.* $\mu(U_n)$ *is computable uniformly in* n*, and* $\lim_n \mu(U_n) = 0$*.*
z *is* Schnorr random *if it passes all Schnorr tests.*

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z *is* Schnorr random *if it passes all Schnorr tests.*

Theorem 10. *TFAE:*

- *A real* z *is Schnorr random.*
- $t(z) < \infty$ *for all c.e. functions* t *such that* $\int t(x)dx$ *is computable.*

Computable function

a base for the topology of $[0, +\infty]$?

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U_i: a computable enumeration of base sets.
$t : [0, 1] \to [0, +\infty]$ is *extended computable* (or *ext-comp.*) if $t^{-1}(U_i) = \{x\ :\ t(x) \in U_i\}$ are c.e. open uniformly in i.

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Remark 11. *by the representation approach*

Kurtz randomness

Definition 12. z *is* Kurtz random *(or* weakly 1-random*) if* $z \in U$ *for every c.e. open set with* $\mu(U) = 1$.

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Theorem 13. *TFAE:*

- *A real z is Kurtz random.*
- *$t(z) < \infty$ for all extended computable functions t such that $\int t(x)dx < \infty$.*
- *$t(z) < \infty$ for all extended computable functions t such that $\int t(x)dx$ is computable.*

Proof

One implication

Proof. (function $\Rightarrow$ test)
$\bigcup_n \{x \ : \ t(x) < n\}$ is a c.e. open set with $\mu(U) = 1$. □

Proof idea

(test $\Rightarrow$ function)
U: a c.e. open set with $\mu(U) = 1$.
Divide U into uniformly c.e. open sets $\{U_n\}_{n\geq 1}$ s.t. $\mu(U_n) = 2^{-n}$.

Proof idea

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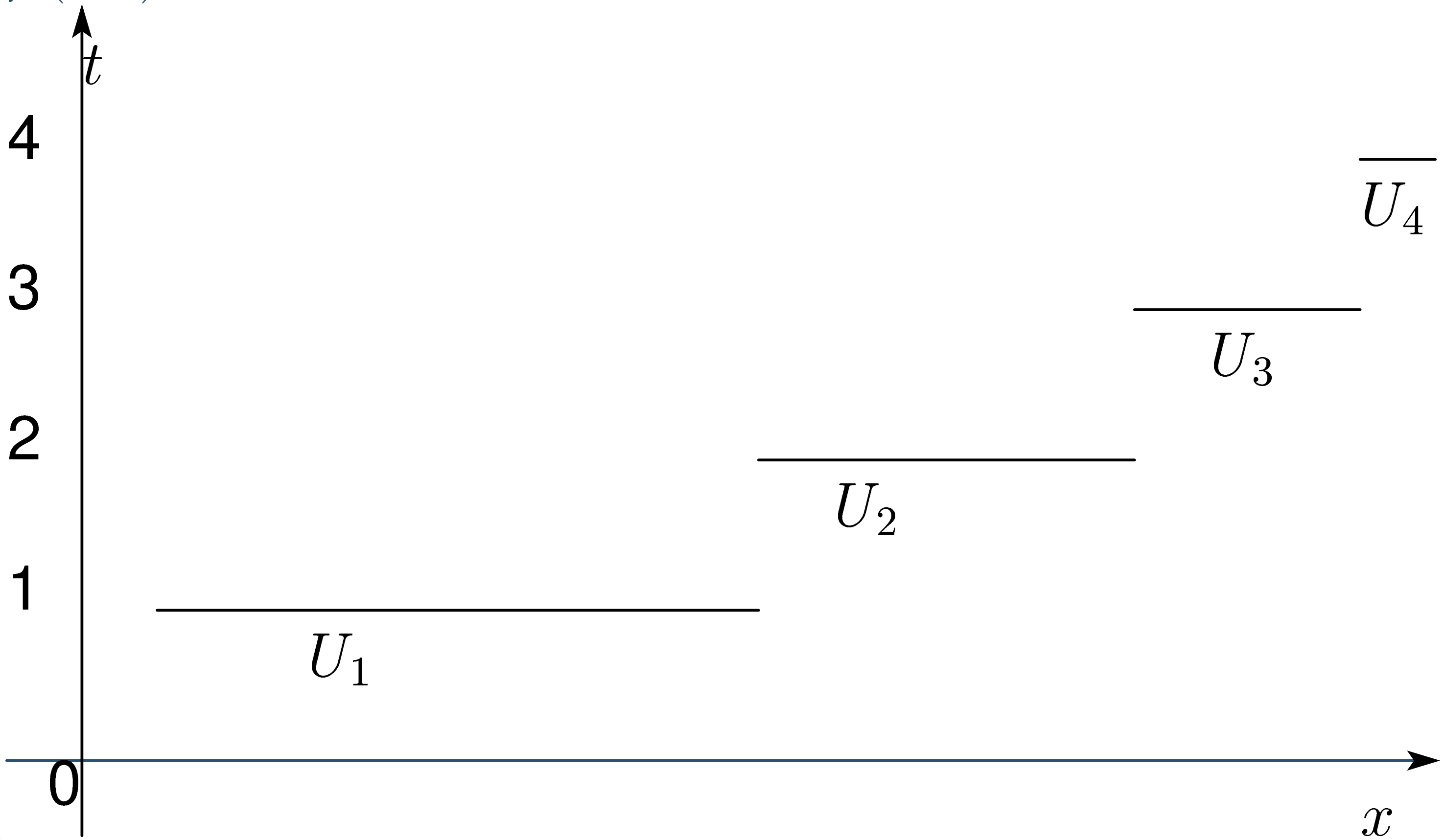

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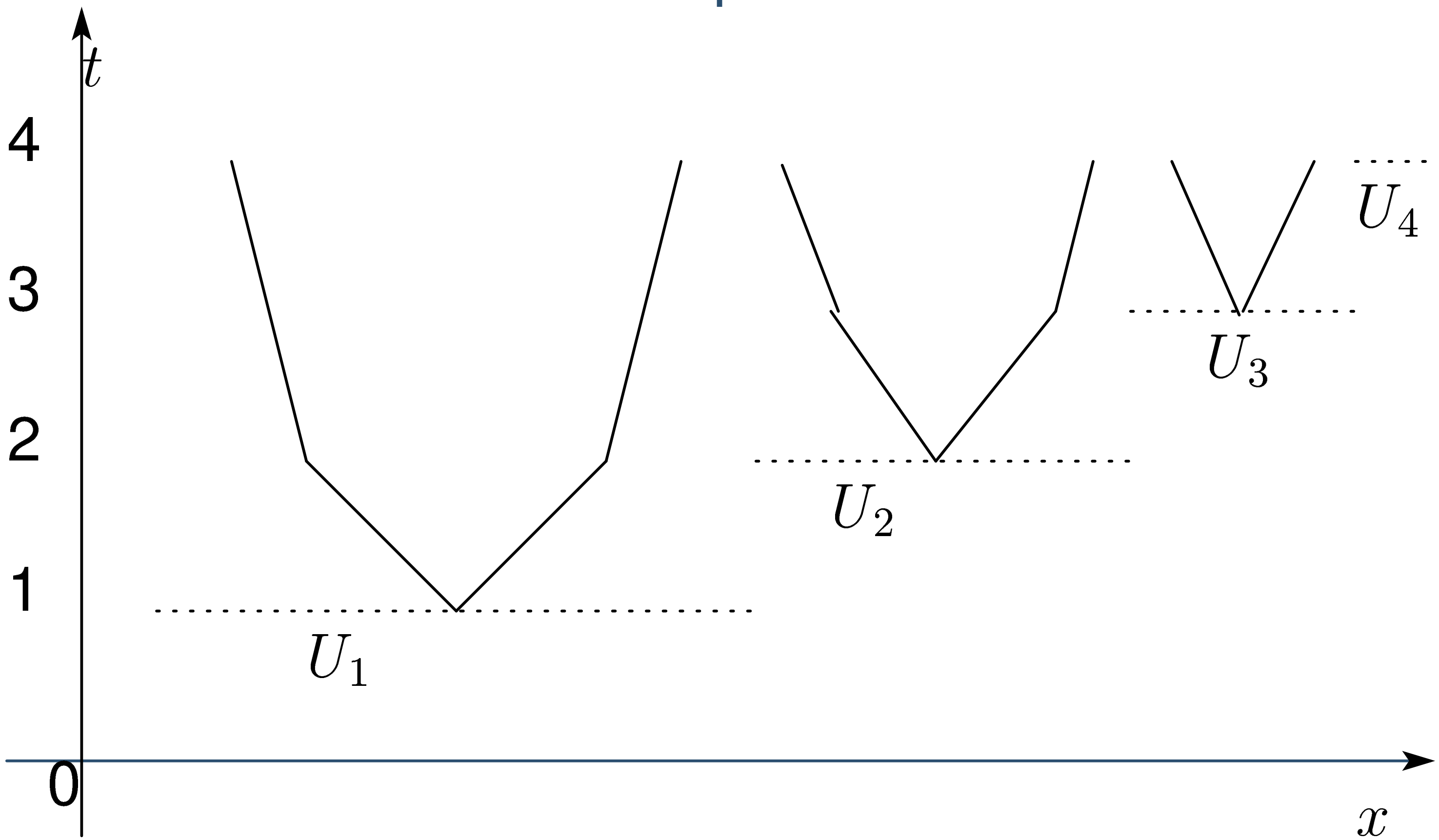

Proof

$g : [0, 1] \to [0, +\infty]$: the polyline s.t.

- the set of endpoints is $\{1 - 2^{-n} : n \geq 0\}$,
- $g(1 - 2^{-n}) = n$,
- $g(1) = \infty$

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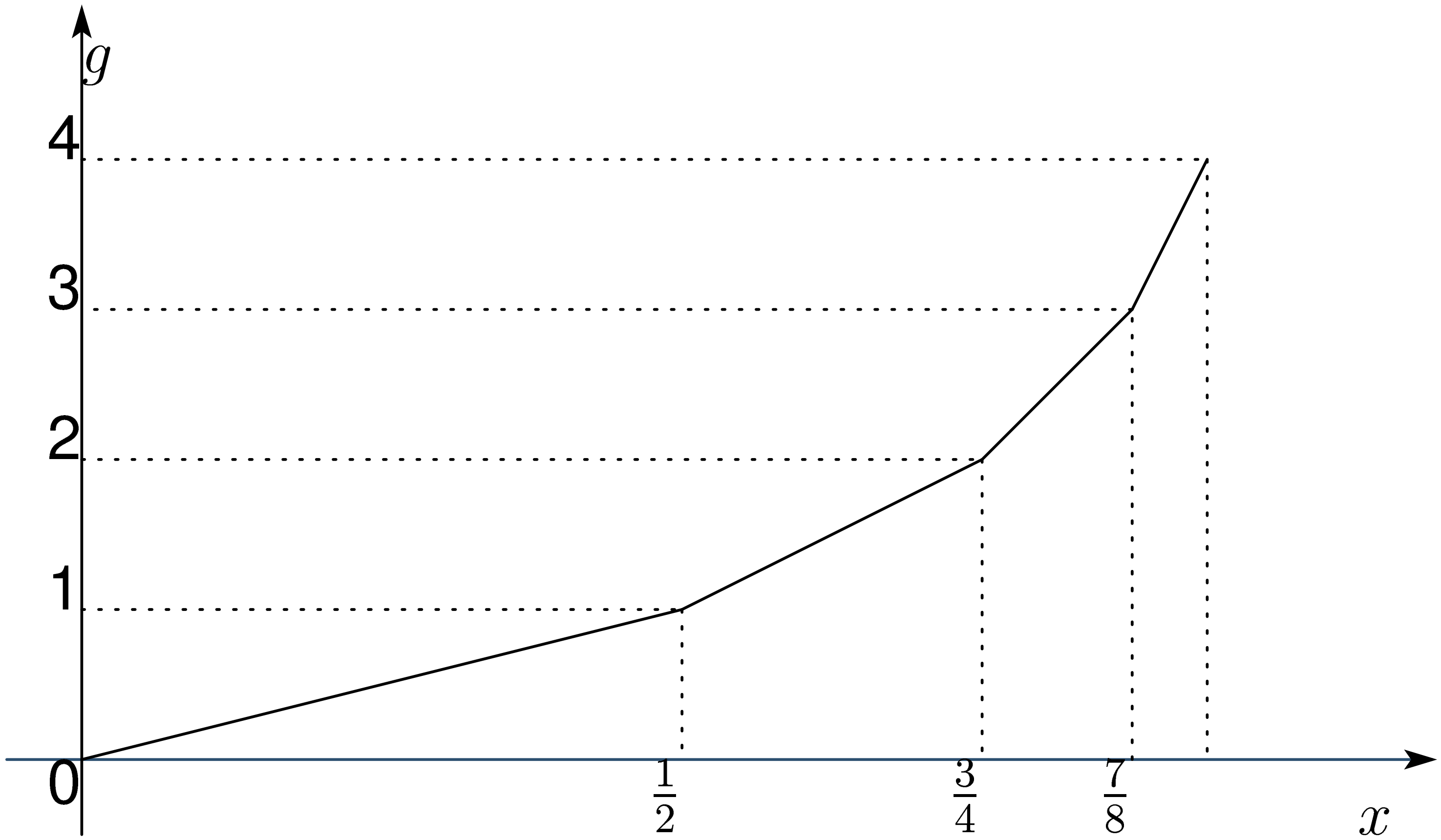
g and the integration are computable.

Proof 2



Proof 3

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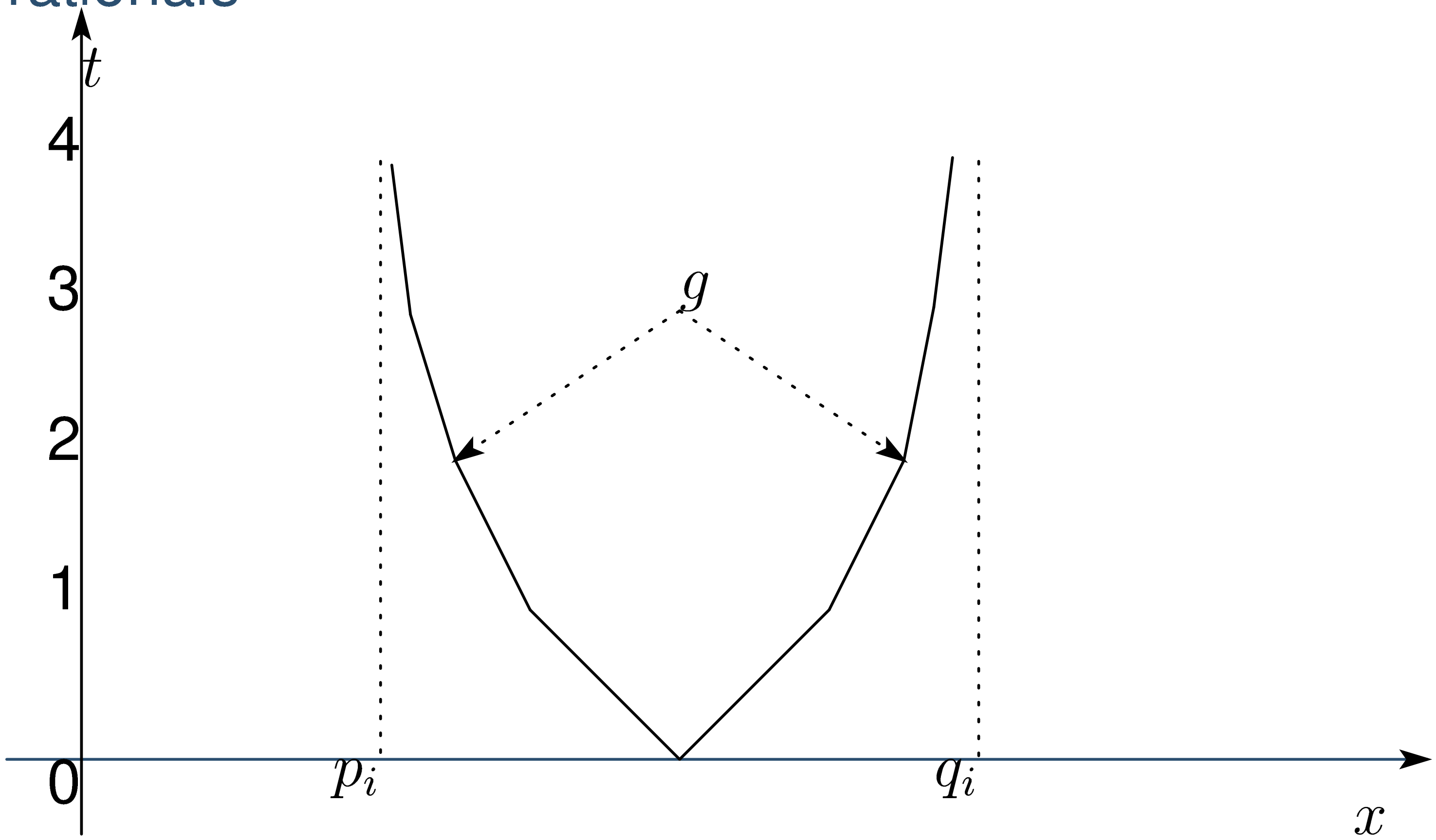

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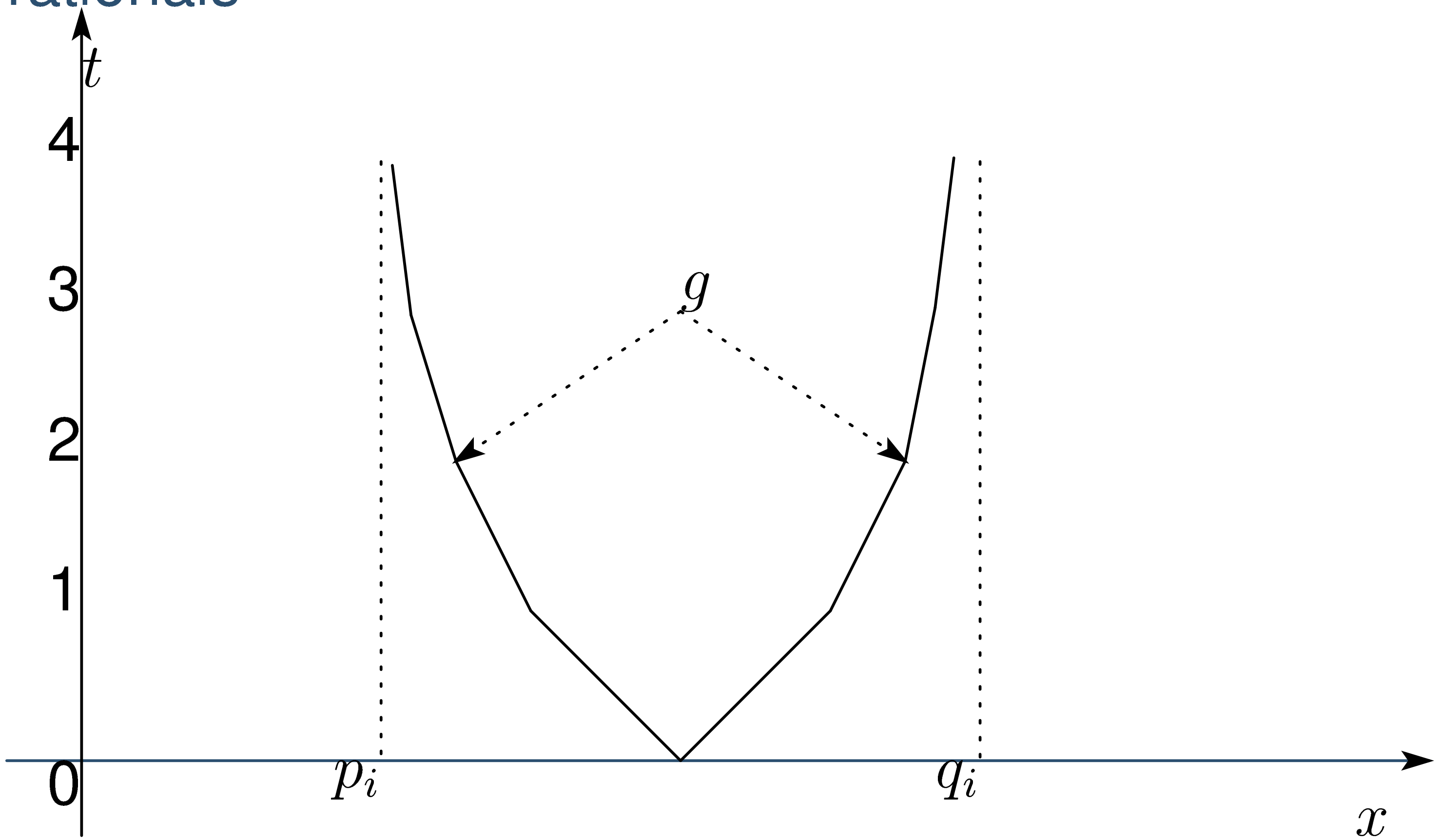

$\int t(x)dx$ is computable.

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The set of pairs (p_i, q_i) is finite for each n
$\Rightarrow t^{-1}([0,n))$

Proof 4

Is t computable?

Are $t^{-1}([0,q)), t^{-1}((p,q)), t^{-1}((p,+\infty])$ uniformly c.e.?

The set of pairs (p_i, q_i) is finite for each n

$\Rightarrow t^{-1}([0,n))$

$\Rightarrow t^{-1}([0,q))$ and $t^{-1}((p,q))$

Proof 5

How to compute $t^{-1}((p, +\infty])$?
Pick up $n \geq p$ and enumerate all pairs (p_i, q_i) until n.

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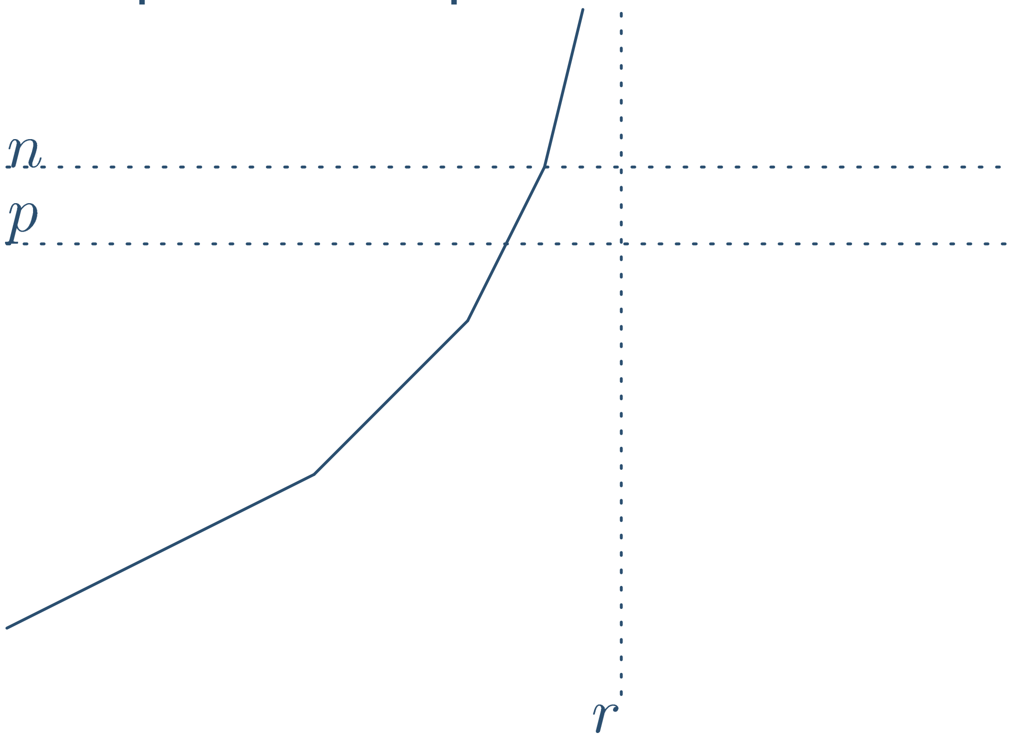

Applications

Remove ”non-negative”

Corollary 14 (by Jason Rute)**.** *TFAE:*

- z *is Kurtz random.*
- $f(z)$ *converges for each a.e. comp. func.*

Remark 15. f, g*: ext-comp. integral tests*
$f - g$ *is an a.e. comp. func.*

Lebesgue Differentiation Theorem

Theorem 16. *TFAE:*

- z *is Kurtz random.*
- $A_r f(z)$ *converges for each ext-comp. integral test.*
- $A_r f(z)$ *converges for each a.e. comp.* L^1*-func.*

$$A_r f(x) = \frac{1}{2r} \int_{[x-r,x+r]} f(x)dx$$

Differentiability

Corollary 17. *TFAE:*

- *A real z is Kurtz random.*
- *f is differentiable at z for each non-dec. comp. func. whose derivative is comp.*
- *f is differentiable at z for each comp. f s.t. f' is a.e. comp.*

Summary for Kurtz randomness

- z is Kurtz random.
- $t(z) < \infty$ for all ext-comp. integral test.
- $f(z)$ converges for each a.e. comp. func. (by Jason Rute)
- $A_r f(z)$ converges for each ext-comp. integral test.
- $A_r f(z)$ converges for each a.e. comp. L^1-func.
- f is differentiable at z for each non-dec. comp. func. whose derivative is comp.
- f is differentiable at z for each comp. f s.t. f' is a.e. comp.

Version of Schnorr randomness

- z is Schnorr random.
- $t(z) < \infty$ for all integral tests with a comp. integration.
- $t(z) < \infty$ for all layerwise comp. s.t. with a comp. integration.
- $f(z) < \infty$ for all L^1-computable functions. (by Hoyrup and Rojas's result)
- $A_r f(z)$ converges for all L^1-computable functions.
- F is differentiable at z for each effectively absolutely continous functions. (by J. Rute)

Version of Martin-Löf randomness

- z is Martin-Löf random.
- $t(z) < \infty$ for all integral tests.
- $A_r f(z)$ converges for all integrable functions s.t. $\int_0^x f(t)dx$ is computable.
- F is differentiable at z for all absolutely continous comp. functions F. (by Freer, Kjos-Hansen and Nies)

Another a.e.

Theorem 18. *z is Kurtz random iff $f(z) = 0$ for each a.e. comp. f with $\int |f(x)|dx = 0$.*

Remark 19. f: continuous with $\int |f(x)|dx = 0$.
$\Rightarrow f(x) = 0$ for all $x \in [0, 1]$.

End

Thank you!