Adapting Rabin's Theorem for Differential Fields

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Computable Fields: the Basics

- A *computable field* F is a field with domain ω, for which the addition and multiplication functions are Turing-computable.
- An element $x \in F$ is *algebraic* if it satisfies some polynomial over the prime subfield $\mathbb{Q}$ or $\mathbb{F}_p$; otherwise x is transcendental. F itself is *algebraic* if all of its elements are algebraic.
- Let $E \models \mathbf{ACF}_0$ be the algebraic closure of F. The *type* over F of an $x \in E$ is determined by its *minimal polynomial* $p(X)$ over F. The formula "$p(X) = 0$" generates a principal type over F iff $p(X)$ is irreducible in $F[X]$. Conversely, every principal 1-type in $\mathbf{ACF}_0$ over F is generated by such a formula.
- $S_F = \{p \in F[X] : (\exists \text{ nonconstant } p_0, p_1 \in F[X])\ p = p_0 \cdot p_1\}$ is the *splitting set* of F. Kronecker showed that $S_{\mathbb{Q}}$ is computable, as is S_F for all finitely generated field extensions F of $\mathbb{Q}$.

Rabin's Theorem, for Fields

Definition

Let F be a computable field. A *Rabin embedding* of F is a computable field embedding $g : F \hookrightarrow E$ such that E is computable, is algebraically closed, and is algebraic over the image $g(F)$.

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Rabin's Theorem (Trans. AMS, 1960)

I. Every computable field F has a Rabin embedding.

II. If $g : F \hookrightarrow E$ is a Rabin embedding, then the following c.e. sets are all Turing-equivalent:

1. The *Rabin image* $g(F)$, within the domain ω of E.
2. The splitting set S_F of F.
3. The *root set* R_F of F:

$$R_F = \{p \in F[X] : (\exists a \in F)\ p(a) = 0\}.$$

Differential Fields

Definition

A *differential field* K is a field with one or more additional unary operations δ satisfying:

$$\delta(x+y) = \delta x + \delta y \quad \text{and} \quad \delta(xy) = x\delta y + y\delta x.$$

K is *computable* if both δ and the underlying field are.

Examples

- The field $\mathbb{Q}(X)$ of rational functions in one variable over $\mathbb{Q}$, with $\delta(y) = \frac{d}{dX}(y)$.
- The field $\mathbb{Q}(X_1, \ldots, X_n)$, with n commuting derivations $\delta_i(y) = \frac{\partial y}{\partial X_i}$.
- Any field, with the trivial derivation $\delta y = 0$.

Every K has a differential subfield $C_K = \{y \in K : \delta y = 0\}$, the *constant field* of K.

Adapting the Notions of Fields

Most field-theoretic concepts have analogues over differential fields.

- $K\{Y\} = K[Y, \delta Y, \delta^2 Y, \delta^3 Y, \ldots]$ is the differential ring of all *differential polynomials* over K.
  Examples of *polynomial differential equations*:
  $$\delta Y = Y, \qquad (\delta^4 Y)^7 - 2Y^3 = 0, \qquad (\delta^4 Y)^3 (\delta Y)^2 Y^8 = 6.$$
  These are ranked according to their *order* and *degree*.
- The theory $\mathbf{DCF}_0$ of *differentially closed fields* was axiomatized by Blum, using:
  $$\forall p, q \in K\{Y\}\ [\mathrm{ord}(p) > \mathrm{ord}(q) \implies \exists x (p(x) = 0 \neq q(x))].$$
  The *differential closure* $\hat{K}$ of K is the prime model of $\mathbf{DCF}_0^K = \mathbf{DCF}_0 \cup \mathrm{AtDiag}(K)$.

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Theorem (Harrington, JSL 1974)

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Theorem (Harrington, JSL 1974)

I. Every computable differential field K has a differential Rabin embedding.

II. ?????

So Harrington proved the first half of Rabin's Theorem for differential fields. However, his proof does not give any insight into what the generators of principal types may be, or what set should be analogous to the splitting set S_F of a field F.

Differential Closures are Different!

If $\text{ord}(p) > 0$, then the equation $p(Y) = 0$ will have infinitely many solutions in the differential closure $\hat{K}$. (If $p(x_1) = \cdots = p(x_n) = 0$, then by Blum, $p(Y) = 0 \neq (Y - x_1)\cdots(Y - x_n)$ has a solution.
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$\hat{K}$ also fails to realize certain 1-types, e.g. the type of an X transcendental over K, but with $\delta X = 0$.

With $K = \mathbb{Q}(X)$, the equation $\delta Y = Y$ certainly has solutions in $\hat{K}$, but the solution $Y = 0$ is different from all the other solutions. All solutions are of the form cy_0, where $c \in K$ with $\delta c = 0$ and $y_0 \neq 0$ is a single fixed solution, and for $c_1 \neq 0 \neq c_2$, the solutions $c_1 y_0$ and $c_2 y_0$ are interchangeable. So the formula "$\delta Y = Y$" does not generate a principal type – but the formula "$\delta Y - Y = 0$ & $Y \neq 0$" does!

Constraints

Definition (from model theory)

For a differential field K, a pair (p,q) from $K\{Y\}$ is a *constraint* if p is monic and algebraically irreducible and $\text{ord}(p) > \text{ord}(q)$ and

$$\forall x, y \in \hat{K}\ [(p(x) = 0 \neq q(x)\ \&\ p(y) = 0 \neq q(y)) \implies x \cong_K y].$$

Facts:

- Every principal type over $\mathbf{DCF}_0^K$ is generated by some constraint. (So every $x \in \hat{K}$ satisfies some constraint.)
- (p,q) is a constraint iff, for all $x, y \in \hat{K}$ satisfying (p,q), x and y are zeroes of exactly the same polynomials in $K\{Y\}$. Thus, being a constraint is Π^0_1.

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Definition

T_K is the set of pairs (p, q) from $K\{Y\}$ which are *not* constraints over K. (So T_K is Σ^0_1, just like S_F.) $\overline{T_K}$ is called the *constraint set*.

Does Rabin's Theorem Carry Over?

Let $g : L \hookrightarrow \hat{K}$ be a Rabin embedding, so $K = g(L)$ is c.e. Assume K is nonconstant. Then the following are computable from an oracle for $T_K (\equiv_T T_L)$:

- K itself, as a subset of $\hat{K}$.
- Algebraic independence over K: the set D_K is decidable:

$$D_K = \{\langle x_1, \ldots, x_n \rangle \in \hat{K}^{<\omega} : \exists h \in K[X_1, \ldots, X_n] \; h(x_1, \ldots, x_n) = 0\}.$$

- The minimal differential polynomial over K of arbitrary $y \in \hat{K}$. This is the unique monic $p \in K\{Y\}$ of least order r and of least degree in $\delta^r Y$ such that $p(y) = 0$. It is the only $p \in K\{Y\}$ for which $\exists q \in K\{Y\}$ [y satisfies (p, q) & $(p, q) \notin T_K$].

So half of Rabin's Theorem holds: $g(L) \leq_T T_L$.

Failure of Rabin's Theorem

Theorem

There exists a computable differential field L with Rabin embedding $g : L \hookrightarrow \hat{L}$ such that $T_L \not\leq_T g(L)$

We set $L_0 = \mathbb{Q}(t_0, t_1, \ldots)$ with $\{t_i\}_{i \in \omega}$ differentially independent over $\mathbb{Q}$. Let g be a Rabin embedding of L_0 into $\hat{L}$, and enumerate $K \supseteq K_0 = g(L_0)$ inside $\hat{L}$ as follows.

1. Set $p_n(Y) = \delta Y - t_n(Y^3 - Y^2)$ (as invented by Rosenlicht).
2. If n enters $\emptyset'$ at stage s, find an $x_n \in \hat{L}$ with $p_n(x_n) = 0$, such that $K_s\langle x_n \rangle \cap \{0, 1, \ldots, s\} \subseteq K_s$. Set $K_{s+1} = K_s\langle x_n \rangle$.

So $n \in \emptyset'$ iff $(p_n, 1) \in T_K$. But each $x \in \hat{L}$ lies in K iff $x \in K_x$, so K is computable. (Moreover, $\hat{L}$ really is a differential closure of K, so the identity map on K is a Rabin embedding into $\hat{K} = \hat{L}$.)

Constrainability

The Rosenlicht polynomials $p_n(Y)$ have another purpose. Let $K_0 = g(L) \subseteq K \subseteq \hat{L}$, still with $L = \mathbb{Q}(t_0, t_1, \ldots)$.

- If $p_n(Y)$ has no zeros in K, then $(p, 1) \in \overline{T_K}$.
- If $p_n(Y)$ has just one zero x_0 in K, then $(p, Y - x_0) \in \overline{T_K}$.
- If $p_n(Y)$ has just two zeros x_0, x_1, then $(p, (Y - x_0)(Y - x_1)) \in \overline{T_K}$.

  $\vdots$

- If p_n has infinitely many zeros in K, then p is *unconstrainable*: there is no $q \in K\{Y\}$ with $(p, q) \in \overline{T_K}$.

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In this last case, what if K contains only half of the (infinitely many) zeros of p_n in $\hat{K}$? The remaining half no longer satisfy any constraint over K. So, although they lie in $\hat{L}$, they fail to lie in $\hat{K}$. That is:

$$g(L) \subsetneq K \subsetneq \hat{K} \subsetneq \hat{L}.$$

Constrainability is Σ^0_2

Recall: $p \in K\{Y\}$ is constrainable over K iff:

$$(\exists q \in K\{Y\})\ (p, q) \in \overline{T_K}.$$

Since $\overline{T_K}$ is Π^0_1, constrainability is Σ^0_2. The same follows from the equivalent condition: p is constrainable iff p is the minimal differential polynomial over K of some $x \in \hat{K}$.

$$(\exists x \in \hat{K})[p(x) = 0 \ \&\ \{x, \delta x, \delta^2 x, \ldots, \delta^{\mathrm{ord}(p)-1} x\} \text{ is alg. indep.}/K].$$

Using Rosenlicht's polynomials, one readily proves:

Theorem

There exists a computable differential field K such that the set of constrainable polynomials in $K\{Y\}$ is Σ^0_2-complete.

A Stronger Result

Theorem

There exists a computable differential field K such that the constraint set $\overline{T_K}$ is Π^0_1-complete and the algebraic dependence set

$$D_K = \{\vec{x} \in K^{<\omega} : (\exists p \in K[\vec{X}])\ p(\vec{x}) = 0\}$$

has high degree $< \mathbf{0}' = \deg(T_K)$.

Proof: We use the same strategy as above to make the set of constrainable polynomials Σ^0_2-complete. Since D_K can enumerate this set, D_K is high. Simultaneously, we code $\emptyset'$ into T_K as before. When we want to enumerate a pair (p_n, q) into T_K, we choose from among infinitely many zeros of $p(Y)$ in $\hat{K}$. This can therefore be mixed with a Sacks preservation strategy, to ensure that D_K cannot compute T_K.

Kronecker's Theorem for Fields

Theorem (Kronecker, 1882)

I. The field $\mathbb{Q}$ has a splitting algorithm (i.e. $S_{\mathbb{Q}}$ is computable).

II. If F has a splitting algorithm and x is algebraic over F, then $F(x)$ has a splitting algorithm, uniformly in the minimal polynomial of x over F.

III. If F has a splitting algorithm and x is transcendental over F, then $F(x)$ has a splitting algorithm.

Parts I and II are crucial for building isomorphisms between algebraic fields. If F has domain $\{x_0, x_1, \ldots\}$, then we find the minimal polynomial of x_0 over $\mathbb{Q}$ (using I), then the minimal polynomial of x_1 over $\mathbb{Q}(x_0)$ (using II), and so on.

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For differential fields, we can now prove the analogue of II. Parts I and III remain open for differential fields. (In I, $\mathbb{Q}$ should be replaced by some simple differential field, such as $\mathbb{Q}(t)$ under $\frac{d}{dt}$.)

Kronecker II: $T_{K\langle z\rangle} \leq_T T_K$

Theorem

For any computable differential field K with nonzero derivation, and any $z \in \hat{K}$, we have $T_{K\langle z\rangle} \leq_T T_K$, uniformly in z.

$\hat{K}$ is also a differential closure of $K\langle z\rangle$, and the identity map on $K\langle z\rangle$ is a Rabin embedding.

$T_{K\langle z\rangle}$ is c.e., so we will show that its complement is c.e. in T_K. Find some $(p_z, q_z) \in \overline{T_K}$ satisfied by z, say of order r_z. Then $K\langle z\rangle = K(z, , \delta z, \ldots, \delta^{r_z - 1} z, \delta^r z)$, and a tuple $\vec{x} \in \hat{K}^{<\omega}$ is algebraically independent over $K\langle z\rangle$ iff $\{\vec{x}, z, \delta z, \ldots, \delta^{r_z - 1} z\}$ is algebraically independent over K, which is decidable in T_K.

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For the proof, we are given (p, q) from $K\langle z\rangle\{Y\}$. The following T_K-computable process halts iff $(p, q) \notin T_{K\langle z\rangle}$.

$(p, q) \notin T_{K\langle z\rangle}$ is $\Sigma_1^{T_K}$

1. Search for $x \in \hat{K}$ with $\{x, \delta x, \ldots, \delta^{\text{ord}(p)-1} x\} \notin D_{K\langle z\rangle}$, such that x satisfies (p, q). Then find $(p_x, q_x) \in \overline{T_K}$ satisfied by x.

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2. Find some $u \in \hat{K}$ such that $K\langle x, z\rangle = K\langle u\rangle$, and find $(p_u, q_u) \in \overline{T_K}$ satisfied by u. Say $u = f(x,z)$, $x = g(u)$, $z = h(u)$.

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3. Let $\tilde{q}(X)$ be the product of the separant and the initial of $p(X)$, the numerator of $q_u(f(X,z))$, and the denominators of $f(X,z)$, $g(f(X,z))$, and $h(f(X,z))$. So $\tilde{q}(x) \neq 0$.

   Fact: If $\tilde{x} \in \hat{K}$ satisfies $(p, \tilde{q})$, then $x \cong_{K\langle z\rangle} \tilde{x}$.

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   Fact: If $\tilde{x} \in \hat{K}$ satisfies $(p, \tilde{q})$, then $x \cong_{K\langle z\rangle} \tilde{x}$.
4. By the Differential Nullstellensatz, we can decide whether $V(p,\tilde{q}) \subseteq V(p,q)$. If so, then every $\tilde{x}$ satisfying (p,q) satisfies $(p,\tilde{q})$, and so $(p,q) \notin T_{K\langle z\rangle}$. If not, then some y satisfies (p,q) but has $\tilde{q}(y) = 0 \neq \tilde{q}(x)$, so $y \ncong_{K\langle z\rangle} x$, and thus $(p,q) \in T_{K\langle z\rangle}$.

$(p, \tilde{q})$ Has the Constraint Property

Suppose $p(\tilde{x}) = 0 \neq \tilde{q}(\tilde{x})$, and set $\tilde{u} = f(\tilde{x}, z)$. Then $q_u(\tilde{u}) \neq 0$.

However, every $j \in K\langle z\rangle\{X\}$ with $j(x) = 0$ has $j(\tilde{x}) = 0$, and we know $p_u(f(x, z)) = 0$. So $\tilde{u}$ satisfies (p_u, q_u), and $u \cong_K \tilde{u}$, say via σ.

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Now $0 = h(u) - z = h(f(x, z)) - z = h(f(\tilde{x}, z)) - z = h(\tilde{u}) - z$, so $\sigma(z) = \sigma(h(u)) = h(\sigma(u)) = h(\tilde{u}) = z$.

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So this σ maps $K\langle z, x\rangle$ isomorphically onto $K\langle z, \tilde{x}\rangle$, fixing $K\langle z\rangle$ and sending x to $\tilde{x}$.

Questions

- What about Kronecker I and III? If z is differentially transcendental over K, must $T_{K\langle z\rangle} \leq_T T_K$? And more importantly: is there a decision procedure for $T_{\mathbb{Q}}$, or for $T_{\mathbb{Q}(a)}$ with $\delta a = 1$?
- Rabin's Theorem for fields showed that $S_F \equiv_T g(F)$. We know that $T_K \equiv g(K)$ fails in general for differential fields. What join of sets or properties of differential fields could be used to replace $g(K)$ and make the statement true? Likewise, what join of sets or properties is $\equiv_T g(K)$?
- Give a more intuitive description of the differential closures of $\mathbb{Q}(a)$, of $\mathbb{Q}(t)$, and of $\mathbb{Q}(t_0, t_1, \ldots)$.