The universality of recursive isomorphism and Borel combinatorics

Andrew Marks

Recursive isomorphism

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If we view x and y as elements of 2^ω, this is equivalent to saying that there's a recursive permutation of the bits of x that yields y.

We analogously define recursive isomorphism between elements of n^ω, or elements of ω^ω.

Countable Borel equivalence relations

Definition

A **Borel equivalence relation** E is an equivalence relation on a Polish space X such that E is Borel as a subset of $X \times X$.

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Most equivalence relations from recursion theory are countable Borel equivalence relations. (Recursive isomorphism, $\equiv_T$, $\equiv_A$, etc.)

Borel reducibility

Definition

If E and F are Borel equivalence relations on X and Y, then E is said to be **Borel reducible** to F iff there is a function $\Phi : X \to Y$ so that for all $x, y \in X$, we have xEy if and only if $\Phi(x)F\Phi(y)$.

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Example:

Folklore

For all reals x, y, we have $x \equiv_T y$ if and only if x' and y' are recursively isomorphic.

Thus, the function $x \mapsto x'$ is a reduction of Turing equivalence to recursive isomorphism.

Borel reducibility for recursive isomorphism

The identify function witnesses

$$
\begin{array}{c}
\text{recursive isomorphism on } \omega^\omega \\
\vdots \\
\rotatebox{90}{$\leq_B$} \\
\text{recursive isomorphism on } 4^\omega \\
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The question reflects a theme in recursion theory where recursion-theoretic structures are often as rich and complicated as possible.

Previous results

Theorem (Dougherty-Kechris, 1991)

Recursive isomorphism on ω^ω is a universal countable Borel equivalence relation

Theorem (Andretta-Camerlo-Hjorth, 2001)

Recursive isomorphism on 5^ω is a universal countable Borel equivalence relation

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Progress on this question

Theorem (M.)

Recursive isomorphism on 3^ω is a universal countable Borel equivalence relation

Whether recursive isomorphism on 2^ω is universal remains open. However, we're able to give a concise explanation of the difference between 2 and 3.

- The reason the proof doesn't generalize to recursive isomorphism on 2^ω is a family of graphs that can be 3-colored, but can't be 2-colored.

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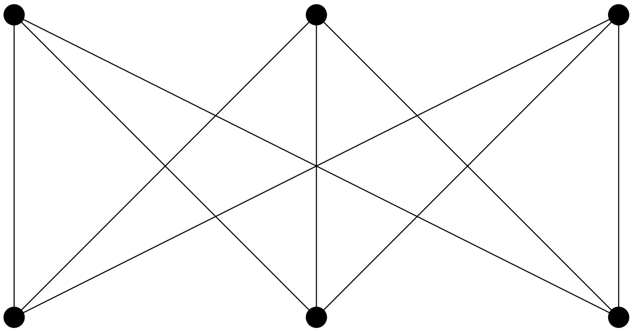
- The reason the proof doesn't generalize to recursive isomorphism on 2^ω is a family of graphs that can be 3-colored, but can't be 2-colored.
- Whether recursive isomorphism on 2^ω is universal seems to be related to a problem in Borel combinatorics.

Some basic notions in combinatorics

A graph on X is a symmetric irreflexive relation on X.

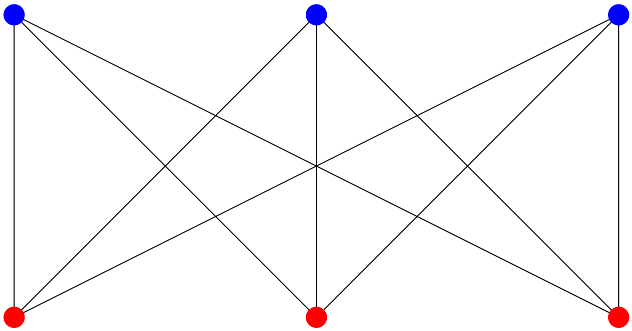
An n-**regular** graph is a graph where each vertex has degree n. A **bipartite** graph is a graph whose vertices can be partitioned into two disjoint sets U and V where no two vertices in U are adjacent, and no two vertices in V are adjacent. The graph drawn below is bipartite 3-regular.



Some basic notions in combinatorics

A graph on X is a symmetric irreflexive relation on X.

A **coloring** of a graph is a function f on the vertices of the graph so that if x and y are neighbors, then $f(x) \neq f(y)$. If the range of f is n, then we say f is an n-coloring.



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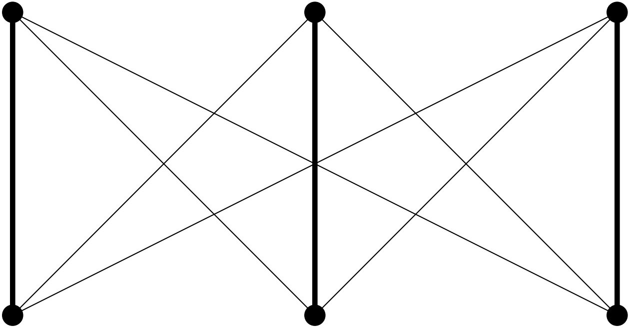
A **matching** of a graph is a subset M of its edges so that each vertex is incident to exactly one edge in M.



Borel combinatorics

The field of Borel combinatorics studies combinatorial problems such as graph colorings, and matchings, but on Borel objects, and where we demand Borel witnesses.

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Definition

A **Borel graph** on 2^ω is a graph G whose vertices are the elements of 2^ω, where the edge relation has a Borel definition. For this talk, all Borel graphs will be on 2^ω.

A **Borel coloring** of a Borel graph G with n colors is a Borel function $c : 2^\omega \to n$ that colors G.

Examples of Borel combinatorics

Classical Theorem (Brooks, 1941)

If G is a graph where each vertex has degree less than or equal to d, then there's a coloring of G with $d+1$ colors

Borel Analogue (Kechris-Solecki-Todorčević, 1999)

If G is a Borel graph where each vertex has degree less than or equal to d, then there's a Borel coloring of G with $d+1$ colors.

Examples of Borel combinatorics

Classical Theorem (König, 1916)

Every bipartite n-regular graph has a Borel perfect matching.

The Borel Analogue is False (Laczkovich, 1988)

There is a Borel bipartite 2-regular graph with no perfect matching.

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Open

Does every Borel bipartite 3-regular graph have a perfect matching?

The combinatorics of the universality of recursive isomorphism

Suppose $\{G_i\}_{i \in \omega}$ is a set of Borel graphs on 2^ω. If $\{c_i\}_{i \in \omega}$ is a set of colorings where c_i colors G_i, say that a point $x \in 2^\omega$ is **monochromatic** if it's assigned the same color by all the c_i.

Open (non-monochromatic 3-coloring)

Suppose $\{G_i\}_{i \in \omega}$ is a countable set of 2-regular Borel graphs. Must there be a set $\{c_i\}_{i \in \omega}$ of Borel 3-colorings of the G_i with no no monochromatic points?

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If non-monochromatic 3-coloring is true, then recursive isomorphism is a universal.

An equivalence

Theorem (M.)

Non-monochromatic 3-coloring is true iff many-1 equivalence is a uniformly universal countable Borel equivalence relation

Uniformly universal means that it is a universal countable Borel equivalence relation, and it also satisfies a uniformity condition in how this universality is witnessed. All known universal equivalence relations are uniformly universal.

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Conjecture

Recursive isomorphism is uniformly universal iff non-monochromatic 3-coloring is true.

An intriguing possibility

Suppose we guess that Borel analogues of classical theorems are rare, and that equivalences from recursion theory are universal iff they're uniformly universal.

Then it seems natural to believe that recursive isomorphism on 2^ω is *not* universal. This would be a striking result – it would say that the recursive isomorphism on 3^ω is fundamentally different from recursive isomorphism on 2^ω, and that this difference doesn't exist between 3^ω and 4^ω, etc.

More about non-monochromatic 3-coloring

- If non-monochromatic 3-coloring is false, this would resolve several open questions in Borel combinatorics in the negative. For instance, this would resolve the open question about Borel perfect matchings.

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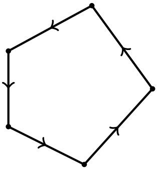
- Non-monochromatic 3-coloring is true for measure and category. That is, you can throw away a meager or null set and find a Borel non-monochromatic 3-coloring on the remaining set.
- This implies that one can't use pure measure or category arguments to show that recursive isomorphism isn't universal.
- Most negative results in Borel combinatorics use measure or category arguments. A counterexample to non-monochromatic 3-coloring would therefore be very interesting since it can't use such techniques.

Where does non-monochromatic 3-coloring come from?

It comes from an ω-length construction. At each stage, we obtain directed graphs consisting of odd length directed cycles. We need to assign 0 or 1 to each vertex. Each time this value changes when we move from a vertex to the next vertex, this corresponds to a real on which we've diagonalized. However, since the cycle has odd length, we can't diagonalize everywhere.

Stage 0:



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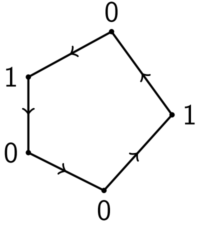

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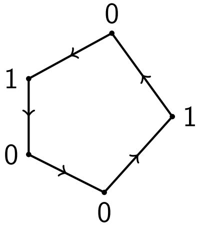

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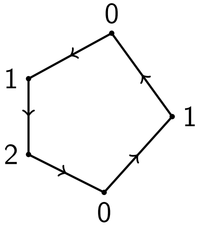

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We get ω many graphs like this, with which we try to diagonalize everywhere. However, if at a single point we use "2" in every coloring, this corresponds to a situation where we never diagonalize on that real.

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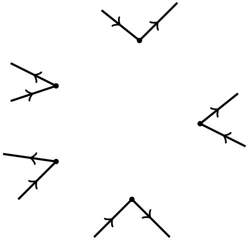

There seems to be a large potential for productive interaction between global problems in recursion theory and Borel combinatorics.