EFFECTIVE FRACTAL DIMENSIONS

Jack H. Lutz

Iowa State University

Lectures

1. Classical and Constructive Fractal Dimensions
2. Dimensions of Points in Euclidean Space
3. Finite-State Dimensions and Polynomial-Time Dimensions

Topics for Today

- Classical fractal dimensions

Topics for Today

- Classical fractal dimensions
- Effectivizing fractal dimensions

Topics for Today

- Classical fractal dimensions
- Effectivizing fractal dimensions
- The constructive (Σ^0_1) case

Topics for Today

- Classical fractal dimensions
- Effectivizing fractal dimensions
- The constructive (Σ^0_1) case
- Dimensions of finite strings

Topics for Today

- Classical fractal dimensions
- Effectivizing fractal dimensions
- The constructive (Σ^0_1) case
- Dimensions of finite strings
- Dimension characterization of Kolmogorov complexity

Topics for Today

- Classical fractal dimensions
- Effectivizing fractal dimensions
- The constructive (Σ^0_1) case
- Dimensions of finite strings
- Dimension characterization of Kolmogorov complexity
- Kolmogorov complexity characterizations of dimension

Euclid, $\approx$ 300 BC : A point is that which has no part.
A line is breadthless length.

Euclid, $\approx$ 300 BC : A point is that which has no part.

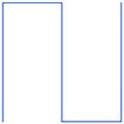
A line is breadthless length.

Points are zero-dimensional.

Curves are one-dimensional.

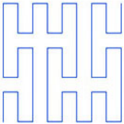
Euclid, $\approx$ 300 BC : A point is that which has no part.
A line is breadthless length.
Points are zero-dimensional.
Curves are one-dimensional.

Peano, et. al., 19th century: Space-filling curves.



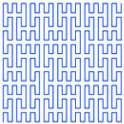
Euclid, $\approx$ 300 BC : A point is that which has no part.
A line is breadthless length.
Points are zero-dimensional.
Curves are one-dimensional.

Peano, et. al., 19th century: Space-filling curves.



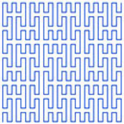
Euclid, $\approx$ 300 BC : A point is that which has no part.
A line is breadthless length.
Points are zero-dimensional.
Curves are one-dimensional.

Peano, et. al., 19th century: Space-filling curves.

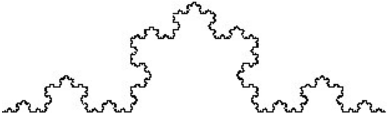


Euclid, $\approx$ 300 BC : A point is that which has no part.
A line is breadthless length.
Points are zero-dimensional.
Curves are one-dimensional.

Peano, et. al., 19th century: Space-filling curves.



Hausdorff, 1919: Rigorous formulation of dimension.



Euclid, $\approx$ 300 BC : A point is that which has no part.
A line is breadthless length.
Points are zero-dimensional.
Curves are one-dimensional.

Caveat (Hausdorff, 1919): Some curves have dimension > 1.

Euclid, $\approx$ 300 BC : A point is that which has no part.
A line is breadthless length.
Points are zero-dimensional.
Curves are one-dimensional.

Caveat (Hausdorff, 1919): Some curves have dimension > 1.

Caveat (today): Some points have dimension > 0.

Euclid, $\approx$ 300 BC : A point is that which has no part.
A line is breadthless length.
Points are zero-dimensional.
Curves are one-dimensional.

Caveat (Hausdorff, 1919): Some curves have dimension > 1.

Caveat (today): Some points have dimension > 0.

.

Euclid, $\approx$ 300 BC : A point is that which has no part.
A line is breadthless length.
Points are zero-dimensional.
Curves are one-dimensional.

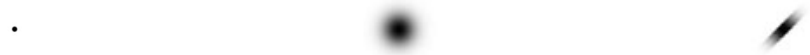
Caveat (Hausdorff, 1919): Some curves have dimension > 1.

Caveat (today): Some points have dimension > 0.

Euclid, $\approx$ 300 BC : A point is that which has no part.
A line is breadthless length.
Points are zero-dimensional.
Curves are one-dimensional.

Caveat (Hausdorff, 1919): Some curves have dimension > 1.

Caveat (today): Some points have dimension > 0.



Euclid, $\approx$ 300 BC : A point is that which has no part.

A line is breadthless length.

Points are zero-dimensional.

Curves are one-dimensional.

Caveat (Hausdorff, 1919): Some curves have dimension > 1.

Caveat (today): Some points have dimension > 0.

Fractal Dimension in Metric Spaces

Let ρ be a metric on a set $\mathcal{X}$.

Fractal Dimension in Metric Spaces

Let ρ be a metric on a set $\mathcal{X}$.

Definition

- The diameter of a set $X \subseteq \mathcal{X}$ is

$$\mathrm{diam}(X) = \sup \{\rho(x, y) \,|\, x, y \in X\}.$$

Fractal Dimension in Metric Spaces

Let ρ be a metric on a set $\mathcal{X}$.

Definition

- The diameter of a set $X \subseteq \mathcal{X}$ is

$$\mathrm{diam}(X) = \sup \{\rho(x, y) \,|x, y \in X\} .$$

- Closed ball of radius r about $x \in \mathcal{X}$:

$$B(x, r) = \{y \in \mathcal{X} \,|\rho(x, y) \leq r\} .$$

Fractal Dimension in Metric Spaces

Let ρ be a metric on a set $\mathcal{X}$.

Definition

- The diameter of a set $X \subseteq \mathcal{X}$ is

$$\mathrm{diam}(X) = \sup \{\rho(x, y) \mid x, y \in X\}.$$

- Closed ball of radius r about $x \in \mathcal{X}$:

$$B(x, r) = \{y \in \mathcal{X} \mid \rho(x, y) \leq r\}.$$

- Open ball

$$B^{o}(x, r) = \{y \in \mathcal{X} \mid \rho(x, y) < r\}.$$

Fractal Dimension in Metric Spaces

Let ρ be a metric on a set $\mathcal{X}$.

Definition

- For $\delta > 0$,

$$\mathcal{C}_\delta = \left\{ \begin{array}{c} \text{countable collections } \mathcal{B} \text{ of balls} \\ \text{with } \mathrm{diam}(B) \leq \delta \text{ for all } B \in \mathcal{B} \end{array} \right\}$$

Fractal Dimension in Metric Spaces

Let ρ be a metric on a set $\mathcal{X}$.

Definition

- For $\delta > 0$,

$$\mathcal{C}_\delta = \left\{ \begin{array}{c} \text{countable collections } \mathcal{B} \text{ of balls} \\ \text{with } \mathrm{diam}(B) \leq \delta \text{ for all } B \in \mathcal{B} \end{array} \right\},$$

$$\mathcal{D}_\delta = \{\text{pairwise disjoint } \mathcal{B} \in \mathcal{C}_\delta\}$$

Fractal Dimension in Metric Spaces

$$\mathcal{C}_\delta = \left\{ \begin{array}{c} \text{countable collections } \mathcal{B} \text{ of balls} \\ \text{with } \mathrm{diam}(B) \le \delta \text{ for all } B \in \mathcal{B} \end{array} \right\}$$

$$\mathcal{D}_\delta = \{\text{pairwise disjoint } \mathcal{B} \in \mathcal{C}_\delta\}$$

Fractal Dimension in Metric Spaces

$$\mathcal{C}_\delta = \left\{ \begin{array}{l} \text{countable collections } \mathcal{B} \text{ of balls} \\ \text{with } \mathrm{diam}(B) \le \delta \text{ for all } B \in \mathcal{B} \end{array} \right\}$$

$$\mathcal{D}_\delta = \{\text{pairwise disjoint } \mathcal{B} \in \mathcal{C}_\delta\}$$

For $X \subseteq \mathcal{X}$ and $\delta > 0$,

$$\mathcal{H}_\delta(X) = \left\{ \mathcal{B} \in \mathcal{C}_\delta \,\middle|\, X \subseteq \bigcup_{B \in \mathcal{B}} B \right\} \qquad (\delta\text{-covers of } X)$$

Fractal Dimension in Metric Spaces

$$\mathcal{C}_\delta = \left\{ \begin{array}{l} \text{countable collections } \mathcal{B} \text{ of balls} \\ \text{with } \operatorname{diam}(B) \leq \delta \text{ for all } B \in \mathcal{B} \end{array} \right\}$$

$$\mathcal{D}_\delta = \{\text{pairwise disjoint } \mathcal{B} \in \mathcal{C}_\delta\}$$

For $X \subseteq \mathcal{X}$ and $\delta > 0$,

$$\mathcal{H}_\delta(X) = \left\{\mathcal{B} \in \mathcal{C}_\delta \,\middle|\, X \subseteq \bigcup_{B \in \mathcal{B}} B \right\}, \qquad (\delta\text{-covers of } X)$$

$$\mathcal{P}_\delta(X) = \{\mathcal{B} \in \mathcal{D}_\delta \,|\, \text{each } B \in \mathcal{B} \text{ is centered in } X\}$$

(δ-packings of X)

Fractal Dimension in Metric Spaces

$$\mathcal{C}_\delta = \left\{ \begin{array}{l} \text{countable collections } \mathcal{B} \text{ of balls} \\ \text{with } \mathrm{diam}(B) \leq \delta \text{ for all } B \in \mathcal{B} \end{array} \right\}$$

$$\mathcal{D}_\delta = \{\text{pairwise disjoint } \mathcal{B} \in \mathcal{C}_\delta\}$$

For $X \subseteq \mathcal{X}$ and $\delta > 0$,

$$\mathcal{H}_\delta(X) = \left\{\mathcal{B} \in \mathcal{C}_\delta \,\middle|\, X \subseteq \bigcup_{B \in \mathcal{B}} B \right\}, \qquad (\delta\text{-covers of } X)$$

$$\mathcal{P}_\delta(X) = \left\{\mathcal{B} \in \mathcal{D}_\delta \,\middle|\, \text{each } B \in \mathcal{B} \text{ is centered in } X \right\}$$

$$(\delta\text{-packings of } X)$$

For $s \geq 0$,

$$H^s_\delta(X) = \inf_{\mathcal{B} \in \mathcal{H}_\delta(X)} \sum_{B \in \mathcal{B}} \mathrm{diam}(B)^s$$

Fractal Dimension in Metric Spaces

$$\mathcal{C}_\delta = \left\{ \begin{array}{l} \text{countable collections } \mathcal{B} \text{ of balls} \\ \text{with } \mathrm{diam}(B) \le \delta \text{ for all } B \in \mathcal{B} \end{array} \right\}$$

$$\mathcal{D}_\delta = \{\text{pairwise disjoint } \mathcal{B} \in \mathcal{C}_\delta\}$$

For $X \subseteq \mathcal{X}$ and $\delta > 0$,

$$\mathcal{H}_\delta(X) = \left\{\mathcal{B} \in \mathcal{C}_\delta \,\middle|\, X \subseteq \bigcup_{B \in \mathcal{B}} B \right\}, \qquad (\delta\text{-covers of } X)$$

$$\mathcal{P}_\delta(X) = \left\{\mathcal{B} \in \mathcal{D}_\delta \,\middle|\, \text{each } B \in \mathcal{B} \text{ is centered in } X \right\}$$

$$(\delta\text{-packings of } X)$$

For $s \ge 0$,

$$H^s_\delta(X) = \inf_{\mathcal{B} \in \mathcal{H}_\delta(X)} \sum_{B \in \mathcal{B}} \mathrm{diam}(B)^s$$

$$P^s_\delta(X) = \sup_{\mathcal{B} \in \mathcal{P}_\delta(X)} \sum_{B \in \mathcal{B}} \mathrm{diam}(B)^s$$

Fractal Dimension in Metric Spaces

$$\mathcal{C}_\delta = \left\{ \begin{array}{l} \text{countable collections } \mathcal{B} \text{ of balls} \\ \text{with } \mathrm{diam}(B) \leq \delta \text{ for all } B \in \mathcal{B} \end{array} \right\}$$

$$\mathcal{D}_\delta = \{\text{pairwise disjoint } \mathcal{B} \in \mathcal{C}_\delta\}$$

For $X \subseteq \mathcal{X}$ and $\delta > 0$,

$$\mathcal{H}_\delta(X) = \left\{\mathcal{B} \in \mathcal{C}_\delta \,\middle|\, X \subseteq \bigcup_{B\in\mathcal{B}} B \right\}, \qquad (\delta\text{-covers of } X)$$

$$\mathcal{P}_\delta(X) = \left\{\mathcal{B} \in \mathcal{D}_\delta \,\middle|\, \text{each } B \in \mathcal{B} \text{ is centered in } X \right\}$$

$$(\delta\text{-packings of } X)$$

For $s \geq 0$,

$$H^s_\delta(X) = \inf_{\mathcal{B}\in\mathcal{H}_\delta(X)} \sum_{B\in\mathcal{B}} \mathrm{diam}(B)^s$$

$$P^s_\delta(X) = \sup_{\mathcal{B}\in\mathcal{P}_\delta(X)} \sum_{B\in\mathcal{B}} \mathrm{diam}(B)^s$$

$$H^s(X) = \lim_{\delta\to 0} H^s_\delta(X)$$

$$P^s_0(X) = \lim_{\delta\to 0} P^s_\delta(X)$$

exist in $[0,\infty]$ by monotonicity

Fractal Dimension in Metric Spaces

$$\mathcal{C}_\delta = \left\{ \begin{array}{l} \text{countable collections } \mathcal{B} \text{ of balls} \\ \text{with } \mathrm{diam}(B) \le \delta \text{ for all } B \in \mathcal{B} \end{array} \right\}$$

$$\mathcal{D}_\delta = \{\text{pairwise disjoint } \mathcal{B} \in \mathcal{C}_\delta\}$$

For $X \subseteq \mathcal{X}$ and $\delta > 0$,

$$\mathcal{H}_\delta(X) = \left\{\mathcal{B} \in \mathcal{C}_\delta \,\middle|\, X \subseteq \bigcup_{B\in\mathcal{B}} B\right\}, \qquad (\delta\text{-covers of } X)$$

$$\mathcal{P}_\delta(X) = \left\{\mathcal{B} \in \mathcal{D}_\delta \,\middle|\, \text{each } B \in \mathcal{B} \text{ is centered in } X\right\}$$

(δ-packings of X)

For $s \ge 0$,

$$H^s_\delta(X) = \inf_{\mathcal{B}\in\mathcal{H}_\delta(X)} \sum_{B\in\mathcal{B}} \mathrm{diam}(B)^s$$

$$P^s_\delta(X) = \sup_{\mathcal{B}\in\mathcal{P}_\delta(X)} \sum_{B\in\mathcal{B}} \mathrm{diam}(B)^s$$

$$H^s(X) = \lim_{\delta\to 0} H^s_\delta(X)$$

$$P^s_0(X) = \lim_{\delta\to 0} P^s_\delta(X)$$

exist in $[0,\infty]$ by monotonicity

$$P^s(X) = \inf\left\{\sum_{i=0}^\infty P^s_0(X_i) \,\middle|\, X \subseteq \bigcup_{i=0}^\infty X_i\right\}$$

optimize over all countable partitions of X

Fractal Dimension in Metric Spaces

$$\mathcal{C}_\delta = \left\{ \begin{array}{l} \text{countable collections } \mathcal{B} \text{ of balls} \\ \text{with } \mathrm{diam}(B) \leq \delta \text{ for all } B \in \mathcal{B} \end{array} \right\}$$

$$\mathcal{D}_\delta = \{\text{pairwise disjoint } \mathcal{B} \in \mathcal{C}_\delta\}$$

For $X \subseteq \mathcal{X}$ and $\delta > 0$,

$$\mathcal{H}_\delta(X) = \left\{\mathcal{B} \in \mathcal{C}_\delta \,\middle|\, X \subseteq \textstyle\bigcup_{B\in\mathcal{B}} B \right\}, \qquad (\delta\text{-covers of } X)$$

$$\mathcal{P}_\delta(X) = \left\{\mathcal{B} \in \mathcal{D}_\delta \,\middle|\, \text{each } B \in \mathcal{B} \text{ is centered in } X \right\}$$

(δ-packings of X)

For $s \geq 0$,

$$H^s_\delta(X) = \inf_{\mathcal{B}\in\mathcal{H}_\delta(X)} \sum_{B\in\mathcal{B}} \mathrm{diam}(B)^s$$

$$P^s_\delta(X) = \sup_{\mathcal{B}\in\mathcal{P}_\delta(X)} \sum_{B\in\mathcal{B}} \mathrm{diam}(B)^s$$

$$H^s(X) = \lim_{\delta\to 0} H^s_\delta(X)$$

$$P^s_0(X) = \lim_{\delta\to 0} P^s_\delta(X)$$

exist in $[0,\infty]$ by monotonicity

$$P^s(X) = \inf\left\{\sum_{i=0}^{\infty} P^s_0(X_i) \,\middle|\, X \subseteq \bigcup_{i=0}^{\infty} X_i \right\}$$

optimize over all countable partitions of X

$H^s(\ \ )$ and $P^s(\ \ )$ are outer measures.

Fractal Dimension in Metric Spaces

$H^s(X)$ = the s-dimensional Hausdorff (outer) ball measure of X
$P^s(X)$ = the s-dimensional packing (outer) ball measure of X

Fractal Dimension in Metric Spaces

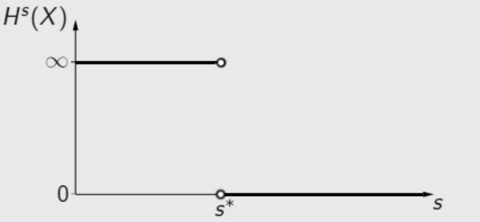
$H^s(X)$ = the s-dimensional Hausdorff (outer) ball measure of X
$P^s(X)$ = the s-dimensional packing (outer) ball measure of X

Definition (Fractal Dimensions)

Let ρ be a metric on $\mathcal{X}$, and let $X \subseteq \mathcal{X}$.

1. (Hausdorff 1919) The Hausdorff dimension of X with respect to ρ is

$$\dim^{(\rho)}(X) = \inf \{ s \,|\, H^s(X) = 0 \} .$$



$H^{s^*}(X)$ can be anything.

Fractal Dimension in Metric Spaces

$H^s(X) =$ the s-dimensional Hausdorff (outer) ball measure of X
$P^s(X) =$ the s-dimensional packing (outer) ball measure of X

Definition (Fractal Dimensions)

Let ρ be a metric on $\mathcal{X}$, and let $X \subseteq \mathcal{X}$.

1. (Hausdorff 1919) The Hausdorff dimension of X with respect to ρ is
$$\dim^{(\rho)}(X) = \inf \{ s \mid H^s(X) = 0 \}.$$
2. (Tricot 1982, Sullivan 1984) The packing dimension of X with respect to ρ is
$$\mathrm{Dim}^{(\rho)}(X) = \inf \{ s \mid P^s(X) = 0 \}.$$

Fractal Dimension in Sequence Spaces

Let Σ be an alphabet with $2 \leq |\Sigma| < \infty$.
A (Borel) probability measure on Σ^∞ is a function $\nu : \Sigma^* \to [0, 1]$ satisfying

$$\nu(\lambda) = 1, \ \nu(w) = \sum_{a \in \Sigma} \nu(wa) \text{ for all } w \in \Sigma^*.$$

Fractal Dimension in Sequence Spaces

Let Σ be an alphabet with $2 \leq |\Sigma| < \infty$.
A (Borel) probability measure on Σ^∞ is a function $\nu : \Sigma^* \to [0, 1]$ satisfying

$\nu(\lambda) = 1$, $\nu(w) = \sum_{a \in \Sigma} \nu(wa)$ for all $w \in \Sigma^*$.

Intuition: Choose $S \in \Sigma^\infty$ "according to ν". Then

$\nu(w) = \mathrm{Prob}[w \sqsubseteq S] = \mathrm{Prob}[w \text{ is a prefix of } S]$.

Fractal Dimension in Sequence Spaces

Let Σ be an alphabet with $2 \leq |\Sigma| < \infty$.
A (Borel) probability measure on Σ^∞ is a function $\nu : \Sigma^* \to [0, 1]$ satisfying

$\nu(\lambda) = 1$, $\nu(w) = \sum_{a \in \Sigma} \nu(wa)$ for all $w \in \Sigma^*$.

Intuition: Choose $S \in \Sigma^\infty$ "according to ν". Then

$\nu(w) = \mathrm{Prob}[w \sqsubseteq S] = \mathrm{Prob}[w \text{ is a prefix of } S]$.

Notation. μ is always the uniform probability measure on Σ^∞, i.e.,

$\mu(w) = |\Sigma|^{-|w|}$.

Fractal Dimension in Sequence Spaces

Let Σ be an alphabet with $2 \leq |\Sigma| < \infty$.
A (Borel) probability measure on Σ^∞ is a function $\nu : \Sigma^* \to [0,1]$ satisfying

$\nu(\lambda) = 1$, $\nu(w) = \sum_{a \in \Sigma} \nu(wa)$ for all $w \in \Sigma^*$.

Intuition: Choose $S \in \Sigma^\infty$ "according to ν". Then

$\nu(w) = \mathrm{Prob}[w \sqsubseteq S] = \mathrm{Prob}[w \text{ is a prefix of } S]$.

Notation. μ is always the uniform probability measure on Σ^∞, i.e.,

$\mu(w) = |\Sigma|^{-|w|}$.

We restrict attention to probability measure that are strongly positive, meaning that there exists $\delta > 0$ such that, for all $w \in \Sigma^*$ and $a \in \Sigma$, $\nu(wa) \geq \delta\nu(w)$.

Fractal Dimension in Sequence Spaces

The metric induced by a strongly positive probability measure ν on Σ^∞ is the function

$$\rho_\nu : \Sigma^\infty \times \Sigma^\infty \to [0, 1]$$

$$\rho_\nu(S, T) = \sup\{\nu(w) \mid w \sqsubseteq S \text{ and } w \sqsubseteq T\}.$$

Fractal Dimension in Sequence Spaces

The metric induced by a strongly positive probability measure ν on Σ^∞ is the function

$$\rho_\nu : \Sigma^\infty \times \Sigma^\infty \to [0,1]$$
$$\rho_\nu(S,T) = \sup\{\nu(w) \mid w \sqsubseteq S \text{ and } w \sqsubseteq T\}.$$

Definition

Let ν be a strongly positive probability measure on Σ^∞, and let $X \subseteq \Sigma^\infty$.

1. The Hausdorff dimension of X with respect to ν is
$$\dim^\nu(X) = \dim^{(\rho_\nu)}(X).$$
2. The packing dimension of X with respect to ν is
$$\mathrm{Dim}^\nu(X) = \mathrm{Dim}^{(\rho_\nu)}(X).$$

Fractal Dimension in Sequence Spaces

The metric induced by a strongly positive probability measure ν on Σ^∞ is the function

$$\rho_\nu : \Sigma^\infty \times \Sigma^\infty \to [0,1]$$
$$\rho_\nu(S,T) = \sup\{\nu(w) \,|\, w \sqsubseteq S \text{ and } w \sqsubseteq T\}.$$

Definition

Let ν be a strongly positive probability measure on Σ^∞, and let $X \subseteq \Sigma^\infty$.

1. The Hausdorff dimension of X with respect to ν is
$$\dim^\nu(X) = \dim^{(\rho_\nu)}(X).$$
2. The packing dimension of X with respect to ν is
$$\mathrm{Dim}^\nu(X) = \mathrm{Dim}^{(\rho_\nu)}(X).$$

$\dim^\nu(X)$ is also called the Billingsley dimension of X (Billingsley 1960).

Fractal Dimension in Sequence Spaces

The metric induced by a strongly positive probability measure ν on Σ^∞ is the function

$$\rho_\nu : \Sigma^\infty \times \Sigma^\infty \to [0,1]$$
$$\rho_\nu(S,T) = \sup\{\nu(w) \,|\, w \sqsubseteq S \text{ and } w \sqsubseteq T\}.$$

Definition

Let ν be a strongly positive probability measure on Σ^∞, and let $X \subseteq \Sigma^\infty$.

1. The Hausdorff dimension of X with respect to ν is
$$\dim^\nu(X) = \dim^{(\rho_\nu)}(X).$$
2. The packing dimension of X with respect to ν is
$$\mathrm{Dim}^\nu(X) = \mathrm{Dim}^{(\rho_\nu)}(X).$$

When $\nu = \mu$, we omit it from the terminology:

- The Hausdorff dimension of X is $\dim_{\mathrm{H}}(X) = \dim^\mu(X)$.
- The packing dimension of X is $\dim_{\mathrm{P}}(X) = \mathrm{Dim}^\mu(X)$.

Gale Characterizations

In a few minutes, we will define martingales, gales, and conditions for their success.

For the moment, martingales are strategies for betting on the successive symbols in a sequence $S \in \Sigma^\infty$, and one of these strategies succeeds on S if it makes an infinite amount of money betting on S.

Gales are generalized martingales that are no more powerful, but exhibit the martingales' success rates in a convenient form.

Gale Characterizations

Ville, 1930's: Martingale success characterizes measure 0 sets.

Gale Characterizations

Ville, 1930's:	Martingale success characterizes measure 0 sets.
Schnorr, 1970's:	Effective martingale success and success rates characterize certain types of randomness.

Gale Characterizations

Ville, 1930's:	Martingale success characterizes measure 0 sets.
Schnorr, 1970's:	Effective martingale success and success rates characterize certain types of randomness.
Lutz, 1990's:	Effective martingale success characterizes measure in complexity classes.

Gale Characterizations

Ville, 1930's:	Martingale success characterizes measure 0 sets.
Schnorr, 1970's:	Effective martingale success and success rates characterize certain types of randomness.
Lutz, 1990's:	Effective martingale success characterizes measure in complexity classes.
Ryabko and Staiger, 1990's:	Effective martingale success rates are related to Hausdorff dimension and Kolmogorov complexity.

Gale Characterizations

Ville, 1930's:	Martingale success characterizes measure 0 sets.
Schnorr, 1970's:	Effective martingale success and success rates characterize certain types of randomness.
Lutz, 1990's:	Effective martingale success characterizes measure in complexity classes.
Ryabko and Staiger, 1990's:	Effective martingale success rates are related to Hausdorff dimension and Kolmogorov complexity.
Lutz, 2000:	Martingale success rates characterize Hausdorff dimension.

Gale Characterizations

Ville, 1930's:	Martingale success characterizes measure 0 sets.
Schnorr, 1970's:	Effective martingale success and success rates characterize certain types of randomness.
Lutz, 1990's:	Effective martingale success characterizes measure in complexity classes.
Ryabko and Staiger, 1990's:	Effective martingale success rates are related to Hausdorff dimension and Kolmogorov complexity.
Lutz, 2000:	Martingale success rates characterize Hausdorff dimension. So used effective martingale success rates to define effective versions of Hausdorff dimension.

Gale Characterizations

Ville, 1930's:	Martingale success characterizes measure 0 sets.
Schnorr, 1970's:	Effective martingale success and <u>success rates</u> characterize certain types of randomness.
Lutz, 1990's:	Effective martingale success characterizes measure in complexity classes.
Ryabko and Staiger, 1990's:	Effective martingale success rates are related to Hausdorff dimension and Kolmogorov complexity.
Lutz, 2000:	Martingale success rates <u>characterize</u> Hausdorff dimension. So used effective martingale success rates to define effective versions of Hausdorff dimension.

Athreya, Hitchcock, Lutz, and Mayordomo 2007:

Same for packing dimension.

Gale Characterizations

Ville, 1930's:	Martingale success characterizes measure 0 sets.
Schnorr, 1970's:	Effective martingale success and success rates characterize certain types of randomness.
Lutz, 1990's:	Effective martingale success characterizes measure in complexity classes.
Ryabko and Staiger, 1990's:	Effective martingale success rates are related to Hausdorff dimension and Kolmogorov complexity.
Lutz, 2000:	Martingale success rates characterize Hausdorff dimension. So used effective martingale success rates to define effective versions of Hausdorff dimension.

Athreya, Hitchcock, Lutz, and Mayordomo 2007:

Same for packing dimension.

Lutz and Mayordomo 2008:

Same for Billingsley dimensions.

Gale Characterizations

Definition

Let ν be a probability measure on Σ^∞, and let $s \geq 0$.

1. A ν-s-gale is a function $d : \Sigma^* \to [0, \infty)$ that satisfies
$$d(w)\nu(w)^s = \sum_{a\in\Sigma} d(wa)\nu(wa)^s$$
for all $w \in \Sigma^*$

Gale Characterizations

Definition

Let ν be a probability measure on Σ^∞, and let $s \geq 0$.

1. A ν-s-gale is a function $d : \Sigma^* \to [0, \infty)$ that satisfies
   $$d(w)\nu(w)^s = \sum_{a\in\Sigma} d(wa)\nu(wa)^s$$
   for all $w \in \Sigma^*$
2. A ν-martingale is a ν-1-gale.

Gale Characterizations

Definition

Let ν be a probability measure on Σ^∞, and let $s \geq 0$.

1. A ν-s-gale is a function $d : \Sigma^* \to [0, \infty)$ that satisfies
$$d(w)\nu(w)^s = \sum_{a \in \Sigma} d(wa)\nu(wa)^s$$
for all $w \in \Sigma^*$
2. A ν-martingale is a ν-1-gale.
3. An s-gale is a μ-s-gale.

Gale Characterizations

Definition

Let ν be a probability measure on Σ^∞, and let $s \geq 0$.

1. A ν-s-gale is a function $d : \Sigma^* \to [0, \infty)$ that satisfies
$$d(w)\nu(w)^s = \sum_{a \in \Sigma} d(wa)\nu(wa)^s$$
for all $w \in \Sigma^*$
2. A ν-martingale is a ν-1-gale.
3. An s-gale is a μ-s-gale.
4. A martingale is a 1-gale.

Gale Characterizations

Observation (Lutz 2000)

d is a ν-s-gale $\iff$ $d'(w) = \nu(w)^{s-1} d(w)$ is a ν-martingale.
$\therefore$ Gales are only a convenience.

Gale Characterizations

Observation (Lutz 2000)

d is a ν-s-gale $\iff$ $d'(w) = \nu(w)^{s-1} d(w)$ is a ν-martingale.
$\therefore$ Gales are only a convenience.

Definition

Let d be a ν-s-gale, and let $S \in \Sigma^\infty$.

1. d succeeds on S if $\limsup_{t\to\infty} d(S[0..t-1]) = \infty$.
2. d succeeds strongly on S if $\liminf_{t\to\infty} d(S[0..t-1]) = \infty$.

Gale Characterizations

Observation (Lutz 2000)

d is a ν-s-gale $\iff$ $d'(w) = \nu(w)^{s-1} d(w)$ is a ν-martingale.
$\therefore$ Gales are only a convenience.

Definition

Let d be a ν-s-gale, and let $S \in \Sigma^\infty$.

1. d succeeds on S if $\limsup_{t \to \infty} d(S[0..t-1]) = \infty$.
2. d succeeds strongly on S if $\liminf_{t \to \infty} d(S[0..t-1]) = \infty$.
3. The success set of d is $S^\infty[d] = \{S \,|\, d \text{ succeeds on } S\}$.
4. The strong success set of d is $S^\infty_{\mathrm{str}}[d] = \{S \,|\, d \text{ succeeds strongly on } S\}$.

Gale Characterizations

Theorem (Lutz & Mayordomo, 2008)

Let ν be a strongly positive probability measure on Σ^∞, and let $X \subseteq \Sigma^\infty$.

1. *The Billingsley ν-dimension of X is*

$$\dim^\nu(X) = \inf \left\{ s \;\middle|\; \begin{array}{l} \text{there is a } \nu\text{-}s\text{-gale } d \\ \text{such that } X \subseteq S^\infty[d] \end{array} \right\}.$$

2. *The strong Billingsley ν-dimension of X is*

$$\mathrm{Dim}^\nu(X) = \inf \left\{ s \;\middle|\; \begin{array}{l} \text{there is a } \nu\text{-}s\text{-gale } d \\ \text{such that } X \subseteq S^\infty_{\mathrm{str}}[d] \end{array} \right\}.$$

Gale Characterizations

Recall

3. The Hausdorff dimension of X is

$$\dim_H(X) = \dim^{\mu}(X).$$

4. The packing dimension of X is

$$\dim_P(X) = \mathrm{Dim}^{\mu}(X).$$

Effective Fractal Dimensions

Everything so far has been classical. Now it's time for the theory of computing.

Effective Fractal Dimensions

Everything so far has been classical. Now it's time for the theory of computing.

Let Δ be a resource bound, such as computable, constructive, poly-time, or finite-state.

Effective Fractal Dimensions

Everything so far has been classical. Now it's time for the theory of computing.

Let Δ be a resource bound, such as computable, constructive, poly-time, or finite-state.
We obtain $\Delta-$effective dimensions by requiring the gales in the gale characterizations to be Δ-computable.

Constructive Dimensions

A very important case is $\Delta =$ constructive.

Constructive Dimensions

A very important case is Δ = constructive.

Definition

A ν-s-gale d is constructive if it is lower semi-computable, i.e., if there is an exactly computable function $\hat{d} : \Sigma^* \times \mathbb{N} \to \mathbb{Q}$ with the following two properties.

- For all $w \in \Sigma^*$ and $t \in \mathbb{N}$, $\hat{d}(w,t) \leq \hat{d}(w,t+1) < d(w)$.
- For all $w \in \Sigma^*$, $\lim_{t\to\infty} \hat{d}(w,t) = d(w)$.

Constructive Dimensions

Definition (Lutz and Mayordomo 2008, aided by a result of Fenner 2002)

Let ν be a strongly positive probability measure on Σ^∞, and let $X \subseteq \Sigma^\infty$.

1. The constructive ν-dimension of X is

$$\mathrm{cdim}^\nu(X) = \inf \left\{ s \;\middle|\; \begin{array}{l} \text{there is a constructive } \nu\text{-}s\text{-gale } d \\ \text{such that } X \subseteq S^\infty[d] \end{array} \right\}$$

2. The constructive strong ν-dimension of X is

$$\mathrm{cDim}^\nu(X) = \inf \left\{ s \;\middle|\; \begin{array}{l} \text{there is a constructive } \nu\text{-}s\text{-gale } d \\ \text{such that } X \subseteq S^\infty_{\mathrm{str}}[d] \end{array} \right\}$$

Constructive Dimensions

Definition (Lutz and Mayordomo 2008, aided by a result of Fenner 2002)

Let ν be a strongly positive probability measure on Σ^∞, and let $X \subseteq \Sigma^\infty$.

1. The constructive ν-dimension of X is
$$\mathrm{cdim}^\nu(X) = \inf \left\{ s \;\middle|\; \begin{array}{l} \text{there is a constructive } \nu\text{-}s\text{-gale } d \\ \text{such that } X \subseteq S^\infty[d] \end{array} \right\}$$
2. The constructive strong ν-dimension of X is
$$\mathrm{cDim}^\nu(X) = \inf \left\{ s \;\middle|\; \begin{array}{l} \text{there is a constructive } \nu\text{-}s\text{-gale } d \\ \text{such that } X \subseteq S^\infty_{\mathrm{str}}[d] \end{array} \right\}$$

We write
$$\mathrm{cdim}(X) = \mathrm{cdim}^\mu(X)$$
and
$$\mathrm{cDim}(X) = \mathrm{cDim}^\mu(X).$$

Constructive Dimensions

A correspondence principle for an effective dimension is a theorem stating that, on sufficiently simple sets, the effective dimension coincides with its classical counterpart. (Terminology stolen from N. Bohr by Lutz.)

Constructive Dimensions

A correspondence principle for an effective dimension is a theorem stating that, on sufficiently simple sets, the effective dimension coincides with its classical counterpart. (Terminology stolen from N. Bohr by Lutz.)

Correspondence Principle for Constructive Dimension

Theorem (Hitchcock 2002)

If $X \subseteq \Sigma^\infty$ is any union (not necessarily effective) of computably closed (i.e., Π^0_1) sets then $\mathrm{cdim}(X) = \dim_{\mathrm{H}}(X)$.

Constructive Dimensions

A correspondence principle for an effective dimension is a theorem stating that, on sufficiently simple sets, the effective dimension coincides with its classical counterpart. (Terminology stolen from N. Bohr by Lutz.)

Correspondence Principle for Constructive Dimension

Theorem (Hitchcock 2002)

If $X \subseteq \Sigma^\infty$ is any union (not necessarily effective) of computably closed (i.e., Π^0_1) sets then $\mathrm{cdim}(X) = \dim_{\mathrm{H}}(X)$.

Correspondence Principle for Constructive Strong Dimension

Constructive Dimensions

A correspondence principle for an effective dimension is a theorem stating that, on sufficiently simple sets, the effective dimension coincides with its classical counterpart. (Terminology stolen from N. Bohr by Lutz.)

Correspondence Principle for Constructive Dimension

Theorem (Hitchcock 2002)

If $X \subseteq \Sigma^\infty$ is any union (not necessarily effective) of computably closed (i.e., Π^0_1) sets then $\mathrm{cdim}(X) = \dim_{\mathrm{H}}(X)$.

Correspondence Principle for Constructive Strong Dimension is false! (Conidis 2009)

Individual Sequences

Definition

Let ν be a probability measure on Σ^∞, and let $S \in \Sigma^\infty$.

1. The ν-dimension of S is $\dim^\nu(S) = \mathrm{cdim}^\nu(\{S\})$.
2. The strong ν-dimension of S is $\mathrm{Dim}^\nu(S) = \mathrm{cDim}^\nu(\{S\})$.

Individual Sequences

Definition

Let ν be a probability measure on Σ^∞, and let $S \in \Sigma^\infty$.

1. The ν-dimension of S is $\dim^\nu(S) = \mathrm{cdim}^\nu(\{S\})$.
2. The strong ν-dimension of S is $\mathrm{Dim}^\nu(S) = \mathrm{cDim}^\nu(\{S\})$.

Absolute Stability of Constructive Dimensions

Theorem (Lutz and Mayordomo 2008, extending Lutz 2000)

If ν is a strongly positive, computable probability measure on Σ^∞, the, for all $X \subseteq \Sigma^\infty$,

$$\mathrm{cdim}^\nu(X) = \sup_{S \in X} \dim^\nu(S)$$

and

$$\mathrm{cDim}^\nu(X) = \sup_{S \in X} \mathrm{Dim}^\nu(S).$$

(Contrast with countable stability of classical dimensions.)

Individual Sequences

Definition

Let ν be a probability measure on Σ^∞, and let $S \in \Sigma^\infty$.

1. The ν-dimension of S is $\dim^\nu(S) = \mathrm{cdim}^\nu(\{S\})$.
2. The strong ν-dimension of S is $\mathrm{Dim}^\nu(S) = \mathrm{cDim}^\nu(\{S\})$.

Absolute Stability of Constructive Dimensions

Theorem (Lutz and Mayordomo 2008, extending Lutz 2000)

If ν is a strongly positive, computable probability measure on Σ^∞, the, for all $X \subseteq \Sigma^\infty$,

$$\mathrm{cdim}^\nu(X) = \sup_{S \in X} \dim^\nu(S)$$

and

$$\mathrm{cDim}^\nu(X) = \sup_{S \in X} \mathrm{Dim}^\nu(S).$$

(Contrast with countable stability of classical dimensions.)

$\therefore$ Constructive dimensions are investigated in terms of individual sequences.

Individual Sequences

In general,

$$\begin{array}{ccccccccc} 0 & \leq & \dim_H(X) & \leq & \dim_P(X) & & \\ & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \\ & & \mathrm{cdim}(X) & \leq & \mathrm{cDim}(X) & \leq & 1. \end{array}$$

Individual Sequences

Definition (Martin-Löf '66, Schnorr '70)

A sequence $R \in \mathcal{C}$ is random if no constructive martingale succeeds on R.

Individual Sequences

If R is random (with respect to the uniform probability measure on $\mathcal{C}$), then

$$\dim(R) = \mathrm{Dim}(R) = 1.$$

Individual Sequences

What if R is random with respect to some other probability measure on $\mathcal{C}$?

Individual Sequences

Fix $\delta > 0$ and a bias sequence $\overrightarrow{\beta} = (\beta_0, \beta_1, \beta_2, \dots)$ with each $\beta_i \in [\delta, 1 - \delta]$.

Definition

$$\mathcal{H}(\beta) = \beta \log \frac{1}{\beta} + (1-\beta) \log \frac{1}{1-\beta} = \text{ Shannon entropy.}$$

$$H_n(\overrightarrow{\beta}) = \frac{1}{n} \Sigma_{i=0}^{n-1} \mathcal{H}(\beta_i)$$

$$H^-(\overrightarrow{\beta}) = \liminf_{n \to \infty} H_n(\overrightarrow{\beta}) \quad \text{lower average entropy}$$

$$H^+(\overrightarrow{\beta}) = \limsup_{n \to \infty} H_n(\overrightarrow{\beta}) \quad \text{upper average entropy}$$

Individual Sequences

Theorem (Athreya, Hitchcock, Lutz, & Mayordomo '07)

Let $0 < \delta \leq \frac{1}{2}$, *and let* $\overrightarrow{\beta} = (\beta_0, \beta_1, \ldots)$ *be a computable bias sequence with each* $\beta_i \in [\delta, \frac{1}{2}]$. *For every* $\overrightarrow{\beta}$*-random sequence* R *we have*

$$\dim(R) = H^{-}(\overrightarrow{\beta}), \ \mathrm{Dim}(R) = H^{+}(\overrightarrow{\beta}).$$

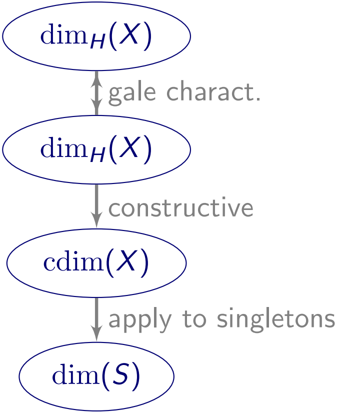
Dimensions of finite strings

Our main task: Extend Hausdorff dimension to define $\dim(x)$ for each $x \in 0, 1^*$.

Dimensions of finite strings

Our strategy:

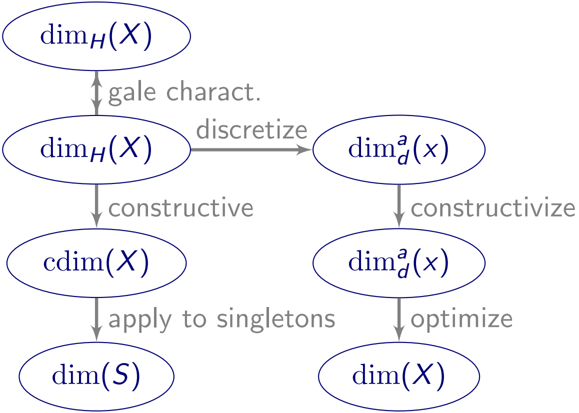
For $X \subseteq \mathcal{C} = \{0,1\}^\infty$



Dimensions of finite strings

Our strategy:

For $X \subseteq \mathcal{C} = \{0,1\}^\infty$ For $x \in \{0,1\}^*$



Dimensions of finite strings

Notation: $\mathcal{T} = \underbrace{\{0,1\}^*}\ \cup\ \underbrace{\{0,1\}^*\square}$

Dimensions of finite strings

Notation: $\mathcal{T} = \underbrace{\{0,1\}^*}_{} \cup \underbrace{\{0,1\}^*\square}_{}$

terminated binary strings

Dimensions of finite strings

Notation: $\mathcal{T} = \underbrace{\{0,1\}^*}_{\substack{\text{prefixes} \\ \text{thereof}}} \cup \underbrace{\{0,1\}^*\square}_{\substack{\text{terminated} \\ \text{binary strings}}}$

Dimensions of finite strings

Definition

An s-termgale is a function $d : \mathcal{T} \to [0, \infty)$ satisfying

$$d(\lambda) \leq 1$$

and

$$d(w) \geq 2^{-s} [d(w0) + d(w1) + d(w\Box)]$$

for all $w \in \{0, 1\}^*$.

Dimensions of finite strings

Definition

An s-termgale is a function $d : \mathcal{T} \to [0, \infty)$ satisfying

$$d(\lambda) \leq 1$$

and

$$d(w) \geq 2^{-s} \left[d(w0) + d(w1) + d(w\Box) \right]$$

for all $w \in \{0, 1\}^*$.

Bets on the successive bits and termination of a finite string.

Dimensions of finite strings

Example

Define $d : \mathcal{T} \to [0, \infty)$ *by*

$$\begin{aligned} d(\lambda) &= 1 \\ d(w0) &= \frac{3}{2} d(w) \\ d(w1) &= d(w\square) = \frac{1}{4} d(w). \end{aligned}$$

Dimensions of finite strings

Example

Define $d : \mathcal{T} \to [0, \infty)$ *by*

$$\begin{aligned} d(\lambda) &= 1 \\ d(w0) &= \frac{3}{2} d(w) \\ d(w1) &= d(w\square) = \frac{1}{4} d(w). \end{aligned}$$

This is a 1-termgale.

Dimensions of finite strings

Example

Define $d : \mathcal{T} \to [0, \infty)$ by

$$\begin{aligned} d(\lambda) &= 1 \\ d(w0) &= \frac{3}{2} d(w) \\ d(w1) &= d(w\Box) = \frac{1}{4} d(w). \end{aligned}$$

This is a 1-termgale. If $w \in \{0,1\}^n$ has n_0 0's and n_1 1's,

$$\begin{aligned} d(w\Box) &= \left(\frac{3}{2}\right)^{n_0} \left(\frac{1}{4}\right)^{n_1+1} \\ &= 2^{n_0(1+\log 3)-2(n+1)} \end{aligned}$$

Dimensions of finite strings

Example

Define $d : \mathcal{T} \to [0, \infty)$ by

$$\begin{aligned} d(\lambda) &= 1 \\ d(w0) &= \frac{3}{2} d(w) \\ d(w1) &= d(w\square) = \frac{1}{4} d(w). \end{aligned}$$

This is a 1-termgale. If $w \in \{0,1\}^n$ has n_0 0's and n_1 1's,

$$\begin{aligned} d(w\square) &= \left(\frac{3}{2}\right)^{n_0} \left(\frac{1}{4}\right)^{n_1+1} \\ &= 2^{n_0(1+\log 3)-2(n+1)} \end{aligned}$$

$$\begin{aligned} \therefore \text{ If } n_0 &>> \frac{2}{1+\log 3}(n+1) \\ &\approx 0.77(n+1), \end{aligned}$$

Dimensions of finite strings

Example

Define $d : \mathcal{T} \to [0, \infty)$ *by*

$$\begin{aligned} d(\lambda) &= 1 \\ d(w0) &= \frac{3}{2} d(w) \\ d(w1) &= d(w\Box) = \frac{1}{4} d(w). \end{aligned}$$

$$\begin{aligned} \therefore \text{ If } n_0 &>> \frac{2}{1 + \log 3}(n+1) \\ &\approx 0.77(n+1), \end{aligned}$$

then $d(w\Box) >> d(\lambda)$, even though d loses $\frac{3}{4}$ of its money when the $\Box$ appears.

Dimensions of finite strings

Trivial observation: If

$$2^{-s|x|}d(x) = 2^{-s'|x|}d'(x) \qquad \bigstar$$

for all $x \in \mathcal{T}$, then d is an s-termgale $\Leftrightarrow$ d' is an s'-termgale.

Dimensions of finite strings

Trivial observation: If

$$2^{-s|x|}d(x) = 2^{-s'|x|}d'(x) \qquad \bigstar$$

for all $x \in \mathcal{T}$, then d is an s-termgale $\Leftrightarrow$ d' is an s'-termgale.
$\therefore$ If d is a 0-termgale, then

$$d'(x) = 2^{s|x|}d(x)$$

is an s-termgale, and all s-termgales are of this form.

Dimensions of finite strings

Trivial observation: If

$$2^{-s|x|}d(x) = 2^{-s'|x|}d'(x) \qquad \bigstar$$

for all $x \in \mathcal{T}$, then d is an s-termgale $\Leftrightarrow$ d' is an s'-termgale.
$\therefore$ If d is a 0-termgale, then

$$d'(x) = 2^{s|x|}d(x)$$

is an s-termgale, and all s-termgales are of this form.

Definition

A termgale is a family

$$d = \left\{ d^{(s)} | s \in [0, \infty) \right\}$$

of s-termgales, one for each s, related by $\bigstar$.

Dimensions of finite strings

Trivial observation: If

$$2^{-s|x|}d(x) = 2^{-s'|x|}d'(x) \qquad \bigstar$$

for all $x \in \mathcal{T}$, then d is an s-termgale $\Leftrightarrow$ d' is an s'-termgale.
$\therefore$ If d is a 0-termgale, then

$$d'(x) = 2^{s|x|}d(x)$$

is an s-termgale, and all s-termgales are of this form.

Definition

A termgale is a family

$$d = \left\{ d^{(s)} | s \in [0, \infty) \right\}$$

of s-termgales, one for each s, related by $\bigstar$.

d is completely determined by any one of its elements.

Dimensions of finite strings

Definition

Let d be a termgale, $a \in \mathbb{Z}^+$, and $w \in \{0,1\}^*$. The dimension of w relative to d at significance level a is

$$\dim_d^a(w) = \inf \left\{ s | d^{(s)}(w\Box) > a \right\}.$$

Dimensions of finite strings

Definition

Let d be a termgale, $a \in \mathbb{Z}^+$, and $w \in \{0,1\}^*$. The dimension of w relative to d at significance level a is

$$\dim_d^a(w) = \inf \left\{ s \middle| d^{(s)}(w\Box) > a \right\}.$$

We have now discretized Hausdorff dimension.

Dimensions of finite strings

Definition

Let d be a termgale, $a \in \mathbb{Z}^+$, and $w \in \{0,1\}^*$. The dimension of w relative to d at significance level a is

$$\dim_d^a(w) = \inf \left\{ s \middle| d^{(s)}(w\Box) > a \right\}.$$

We have now discretized Hausdorff dimension. Constructivizing is easy:

Definition

A termgale d is constructive if $d^{(0)}$ is lower semicomputable.

Dimensions of finite strings

Now optimize:

Definition

A constructive termgale $\tilde{d}$ is optimal if for every constructive termgale d there is a constant $\alpha > 0$ such that, for all $s \in [0, \infty)$ and $w \in \{0, 1\}^*$,

$$\tilde{d}^{(s)}(w\Box) \geq \alpha d^{(s)}(w\Box).$$

Dimensions of finite strings

Now optimize:

Definition

A constructive termgale $\tilde{d}$ is <u>optimal</u> if for every constructive termgale d there is a constant $\alpha > 0$ such that, for all $s \in [0, \infty)$ and $w \in \{0,1\}^*$,

$$\tilde{d}^{(s)}(w\Box) \geq \alpha d^{(s)}(w\Box).$$

Theorem (Lutz 2003)

If $\tilde{d}$ is an optimal constructive termgale, then, for every constructive termgale d and every $a \in \mathbb{Z}^+$, there is a constant $\gamma \in [0, \infty)$ such that, for all $w \in \{0,1\}^*$,

$$\dim_{\tilde{d}}^{a}(w) \leq dim_d(w) + \frac{\gamma}{1 + |w|}.$$

Dimensions of finite strings

Corollary

If d_1, d_2 are optimal constructive termgales and $a_1, a_2 \in \mathbb{Z}^+$, then there is a constant $\alpha \in [0, \infty)$ such that, for all $w \in \{0,1\}^*$,

$$\left|\dim_{d_1}^{a_1}(w) - \dim_{d_2}^{a_2}(w)\right| \leq \frac{\alpha}{1 + |w|}.$$

Dimensions of finite strings

Corollary

If d_1, d_2 are optimal constructive termgales and $a_1, a_2 \in \mathbb{Z}^+$, then there is a constant $\alpha \in [0, \infty)$ such that, for all $w \in \{0,1\}^*$,

$$\left|\dim_{d_1}^{a_1}(w) - \dim_{d_2}^{a_2}(w)\right| \leq \frac{\alpha}{1 + |w|}.$$

$\therefore$ It makes very little difference which optimal constructive termgale or which significance level we use.

Dimensions of finite strings

Corollary

If d_1, d_2 are optimal constructive termgales and $a_1, a_2 \in \mathbb{Z}^+$, then there is a constant $\alpha \in [0, \infty)$ such that, for all $w \in \{0,1\}^*$,

$$\left|\dim_{d_1}^{a_1}(w) - \dim_{d_2}^{a_2}(w)\right| \leq \frac{\alpha}{1 + |w|}.$$

$\therefore$ It makes very little difference which optimal constructive termgale or which significance level we use.

Theorem (Lutz 2003)

There is an optimal constructive termgale $\mathbf{d}_{\square}$.

Proof uses Levin's $\mathbf{m}$.

Dimensions of finite strings

Corollary

If d_1, d_2 are optimal constructive termgales and $a_1, a_2 \in \mathbb{Z}^+$, then there is a constant $\alpha \in [0, \infty)$ such that, for all $w \in \{0,1\}^*$,

$$\left|\dim_{d_1}^{a_1}(w) - \dim_{d_2}^{a_2}(w)\right| \leq \frac{\alpha}{1 + |w|}.$$

$\therefore$ It makes very little difference which optimal constructive termgale or which significance level we use.

Theorem (Lutz 2003)

There is an optimal constructive termgale $\mathbf{d}_\square$.

Proof uses Levin's $\mathbf{m}$.

Definition

The dimension of a string $w \in \{0,1\}^*$ is

$$\dim(w) = \dim_{\mathbf{d}_\square}^{1}(w).$$

Dimension and Kolmogorov Complexity

Theorem (Lutz 2003)

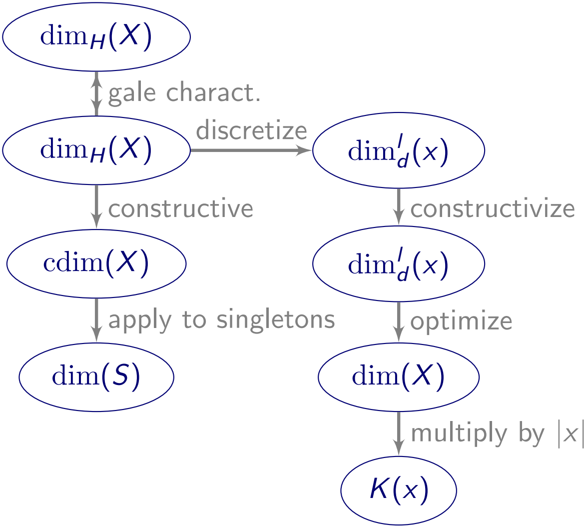
There is a constant $c \in \mathbb{N}$ *such that, for all* $x \in 0,1^*$,

$$|K(x) - |x|\mathrm{dim}(x)| \leq c.$$

Dimension and Kolmogorov Complexity

Our strategy:

For $X \subseteq \mathcal{C} = \{0,1\}^\infty$ For $x \in \{0,1\}^*$



Dimension and Kolmogorov Complexity

$\therefore$ Up to constant additive terms,

$$K(x) = \log \frac{1}{\mathbf{m}(x)} = |x| \dim(x).$$

Three fundamentally different approaches to information, when constructivized and optimized (after discretizing $\dim_H$) lead to the same fundamental quantity, $K(x)$.

Kolmogorov Complexity Characterizations

Our dimension characterization of $K(\alpha)$ is one of several ways to prove the following.

Theorem (Lutz and Mayordomo 2008)

If ν is a strongly positive, computable probability measure on Σ^∞, then, for all $S \in \Sigma^\infty$,

$$\dim^\nu(S) = \liminf_{m\to\infty} \frac{\mathrm{K}(S[0..m-1])}{\mathcal{I}_\nu(S[0..m-1])}$$

and

$$\mathrm{Dim}^\nu(S) = \limsup_{m\to\infty} \frac{\mathrm{K}(S[0..m-1])}{\mathcal{I}_\nu(S[0..m-1])},$$

where

$\mathrm{K}(w) =$ *Kolmogorov complexity of w (any flavor)*

and

$\mathcal{I}(w) =$ *Shannon ν-self-information of w* $= \log \frac{1}{\nu(w)}$.

Thank you!