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Honesty and Time-Constructibility in Type-2 Computation

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Type-2 Computations (machine model)

$$F: (N \to N) \times N \to N \qquad F(f,x) = \sum_{i=0}^{x} f(i)$$

$$g(x) = x \qquad F(g,3) = 0 + 1 + 2 + 3 = 6$$

Based on an Oracle Truing Machine Model for type-2 computations, we have

- Compression theorem (Inflation theorem 2001, Li & Royer)
- Speedup theorem (CiE 2007)
- Union theorem (CiE 2009)
  - Do we have a gap theorem?
  - Yes

Do we have a honesty theorem?

- Yes
- But we don’t need it.
- The definition of the bounds captures the honesty property.

Complexity classes (J. Hartmanis & R. Stearns 1965)

resource bound

$$C(t) = \{f \in R \mid \exists e[\varphi_e = f, \forall^\infty x(\Phi_e(x) \leq t(|x|))]\}$$

Most natural complexity class can be better understood as a union of some classes defined as above.

$$PTIME = \bigcup_{k \in N} DTIME(n^k)$$

$$C(t) \mathrel{?=} PTIME \qquad \text{for some } t$$

Blum's two axioms, 1967

$\varphi_{i \in N}$ Acceptable programming system

$\Psi_{i \in N}$ Blum complexity measure

Axiom 1: $\varphi_i(x) \downarrow \Leftrightarrow \Psi_i(x) \downarrow$

Axiom 2: $\Psi_i(x) = m$ is decidable

Union Theorem (E. McCreight & A. Meyer, 1969)

Given any sequence of recursive functions $f_0, f_1, f_2, \cdots$
such that,

$\lambda i, x. f_i(x)$ is recursive

and for all $i, x \in N, f_i(x) \leq f_{i+1}(x)$

then, there is a recursive function *g* such that $C(g) = \bigcup_{i \in N} C(f_i)$

$$PTIME = \bigcup_{k \in N} DTIME(n^k) \qquad C(t) = PTIME$$

Blum's Measured Sets, 1967

M is a measured set, if there is a recursive *g* such that

$$\mathrm{M} = \{\varphi_{g(i)} \mid i \in N\}$$

and each function in M is a complexity measure.

The Compression theorem (Blum '67)

Given any measured set, we can uniformly increase every complexity class defined by some complexity measure in the set.

That is, $C(\varphi_{g(i)}) \subset C(\varphi_{k(i)})$

Gap Theorem (Borodin,1969)

For any recursive function *r*, there is a recursive *t* such that

$$C(t) = C(r \circ t)$$

Operator Gap Theorem (Constable 1972)

$$C(t) = C(\Theta(t))$$

Why Honesty Theorem?

- If we don't like the gap, we need to restrict our bounds to a measured set.
- Any function in a measured set is honest (*g*-honest)

$$\mathrm{M} = \{\varphi_{g(i)} \mid i \in N\}$$

- Do we lose any complexity class to be measured? No

I.e., Is there a recursive function t such that, for all i

$$C(t) \neq C(\varphi_{g(i)})$$

Honesty Theorem (McCreight & Meyer 1969)

- There is a recursive function g that determined a measured set

$$\mathrm{M} = \{\varphi_{g(i)} \mid i \in N\}$$

such that, for every recursive function f such that, for all i , if $f = \varphi_i$

$$C(f) = C(\varphi_{g(i)})$$

The same questions at type-2:

OTM: Oracle Turing Machine

What is the resource bound for type-2 computation?

Should the bound be type-1 or type-2?

A natural type-2 analog to PTIME (S. Cook & B. Kapron 1989)

$$\mathbf{C}(\boldsymbol{p}) = \left\{ \widehat{\varphi}_e \;\middle|\; \forall (f, x) \in \mathcal{T} \times \mathbf{N} \left[\widehat{\Phi}_e(f, x) \leq \boldsymbol{p}(|f|, |x|) \right] \right\}.$$

Basic Feasible Functional (BFF) at type-2.

$$BFF = \bigcup_{p \in P} C(p)$$

Type-2 Time Bounds (T_2TB):

Dynamic resource bound for type-2 computations

$$\beta : F \times N \rightarrow N$$

F, the set of finite functions

1. Computable
2. Nontrivial
3. Bounded
4. Convergent
5. *F*-monotone (optional for strong bounds)

- **Some appropriate clocking scheme**
- **Some appropriate definition of small sets (compact)**

Then, $C(\beta)$ is a workable notion for type-2 complexity classes.

Type-2 Gap Theorem

For any recursive function r, there is a T_2TB β such that

$$C(\beta) = C(r \circ \beta)$$

Note:

- This theorem is not very robust; it is very sensitive to the cost of handling oracle answers.
- We can use the same idea (measured sets) to remove the gap phenomena.
- Then, do we have a type-2 honesty theorem?

Type-2 Honesty Theorem

- There is a recursive function g that determined a measured set

$$M = \{\varphi_{g(i)} \mid i \in N\}$$

such that, for every β in T_2TB , such that, for all i , if $\beta = \varphi_i$, then

$$C(\beta) = C(\varphi_{g(i)})$$

Note: However, we don’t think this theorem is interesting since the gap theorem in the previous slide is not interesting.

For any effective operator Θ, can we always find *β* such that Θ can't enhance *β*?

$$C(\beta) = C(\Theta(\beta))$$

Note:

- We no longer can have a free ride from the classical type-1 theorem.

- Since an arbitrary effective operator may not result in a well defined T_2TB.

For any T_2TB β, there always exists some effective operator $\Theta: T_2TB \rightarrow T_2TB$ such that

$$\mathcal{C}(\beta) \subsetneqq \mathcal{C}(\Theta(\beta)) \subsetneqq \mathcal{C}(\Theta^2(\beta)) \subsetneqq \ldots$$

Thank you!