Multi-Resolution Cellular Automata for Real Computation

James I. Lathrop, Jack H. Lutz, and Brian Patterson

Iowa State University Department of Computer Science

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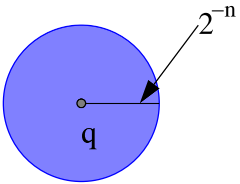
Introduction

1. Prior Work
    - Bit Computability
    - Cellular Automata
2. Computably Open Sets and Computable Sets
3. Multi-Resolution Cellular Automata and Results

Bit Computability: Introduction

Computability of sets in Euclidean space as defined by Braverman [2]:

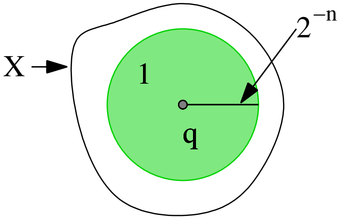
- For $q \in \mathbb{Q}^2$ and $n \in \mathbb{N}$, let $B(q, 2^{-n})$ denote the open ball of radius 2^{-n} with center q.



- A set $X \subseteq \mathbb{R}^2$ is (bit) *computable* if there is a computable function $f : \mathbb{Q}^2 \times \mathbb{N} \to \{0, 1\}$ such that
  - (i) If $B(q, 2^{-n}) \subseteq X$, then $f(q, n) = 1$ (i.e. turns green).
  - (ii) If $B(q, 2^{1-n}) \cap X = \emptyset$, then $f(q, n) = 0$ (i.e. turns red).

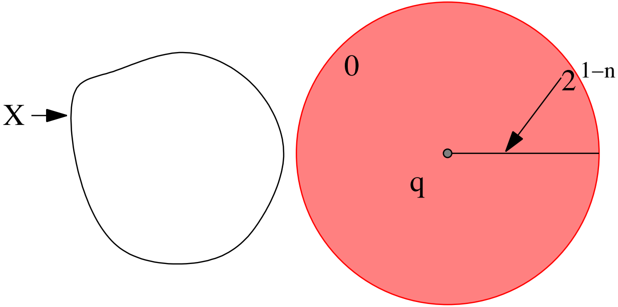
Bit Computability: Accepting

(i) If $B(q, 2^{-n}) \subseteq X$, then $f(q, n) = 1$ (i.e. turns green).



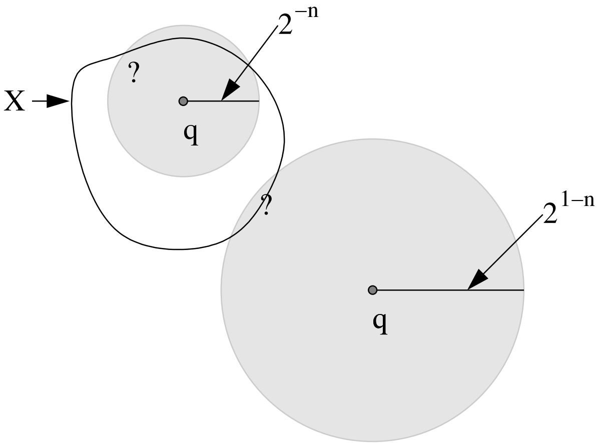
Bit Computability: Rejecting

(ii) If $B(q, 2^{1-n}) \cap X = \emptyset$, then $f(q, n) = 0$ (i.e. turns red).



Bit Computability: ?

- Note: If the hypotheses of (i) and (ii) are both false, then $f(q, n)$ must still be defined and may be either 1 or 0.



Bit Computability: Strengths and Weaknesses

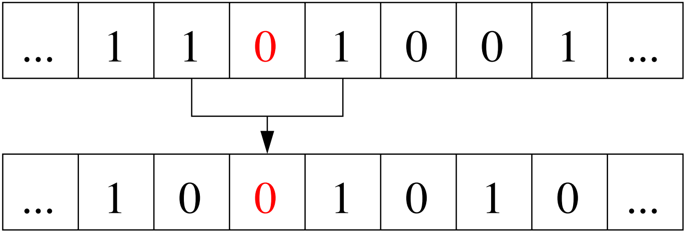
Advantages:

- Succinct and rigorous.
- Intuitively akin to the standard definition of a limit:
  - Limit: $\lim_{n \to \infty} x_n = L$ defined in terms of ϵ and n_0.
  - Computability: $X \subseteq \mathbb{R}^2$ defined in terms of $q \in \mathbb{Q}^2$ and n where, as n increases, we approach an exact characterization of X.

Disadvantages:

- Requires a q and an n to evaluate any set of points.
  - What if we instead take a cue from non-standard analysis and just keep generating approximations?
- It would also be nice to have a spatial model of computation to decide sets of real numbers.

Forming the basis of our model is the Cellular Automaton (CA).



$$\delta : \{0,1\} \times \{0,1\} \times \{0,1\} \rightarrow \{0,1\}$$

Example: $\delta(1,0,1) \rightarrow 0$

Cellular Automata: Choices

Choices made for our basic CA:

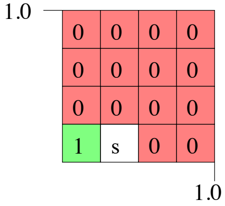
- Two-dimensional
- North, South, East, West immediate neighbors
- More expansive state set than just 0 and 1

Cellular Automata: Simple Modifications

Simple changes to CA:

- "Color" accept and reject state(s) green and red
- Restrict ourselves to the unit square over $\mathbb{R}^2$
- Assign each cell a partition of that space to represent

Example:

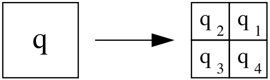


Multi-Resolution Cellular Automata (MRCA)

Main alteration of a CA to create a MRCA:

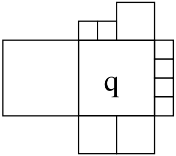
Fissions may replace transitions in the rule set.

- Allow each cell to fission into subcells instead of transitioning.
- Subcells replace the parent, have the same properties and method of operation (just on a smaller scale).
- Number of subcells specified by: dimension of the MRCA and splitting factor.

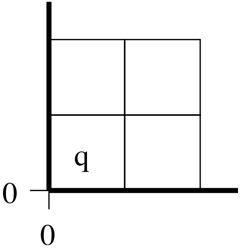


When can a cell **not** read its neighbor?

- Neighbor is too small.



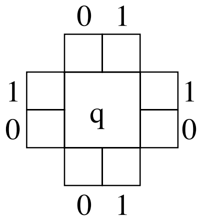
- Neighbor does not exist.



MRCA: Modelling Issues

Our solution:

- A cell can read neighbors up to 1 size smaller.
  - Use index to differentiate north 0, north 1, east 0, east 1, etc.



- A neighbor of the same size or larger gives the same result for both indices.
- Unable to read? A $\perp$ is read.

MRCA Computable: Two Useful Definitions

To formally define MRCA computable, we use the following terms:

- If X is *computably open*, then there exists an algorithm to computably enumerate open balls (or cells) $\mathcal{A}$ such that $X = \cup\mathcal{A}$.
- If X is *dense on* Y, then the closure of X is Y.

MRCA Computable

A set $X \subseteq [0,1]^2$ is *MRCA computable* if:

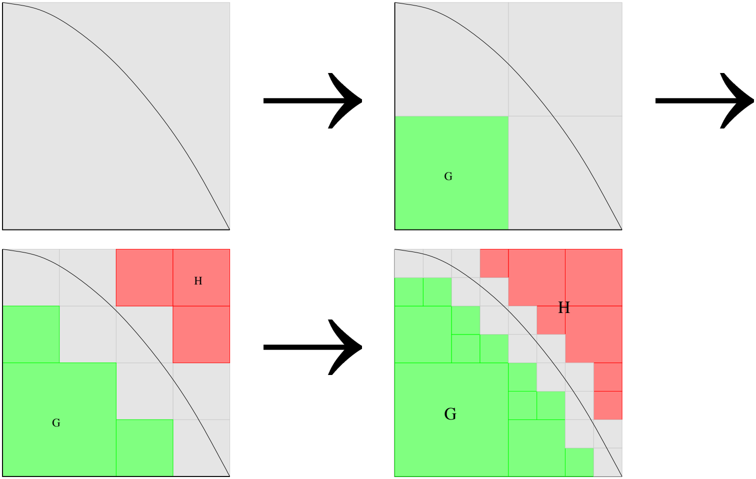
There exists computably open sets $G \subseteq X$ and $H \subseteq [0,1]^2 \setminus X$, with $G \cup H$ dense on $[0,1]^2$, and an MRCA that, starting with all cells uncolored, achieves the following.

(I) For every $x \in G$, there is some finite time at which x (i.e. some cell containing x) becomes green and stays that way.

(II) For every $x \in H$, there is some finite time at which x (i.e. some cell containing x) becomes red and stays that way.

MRCA Computable Example

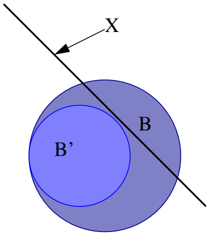
Example MRCA computing of $\{(x, y) \mid y < 1 - x^2\}$.



Bit and MRCA Computable: Two Definitions to Relate

How does MRCA computability related to bit computability?

- A set X is *computably nowhere dense* if there is a computable function f that, given any ball B, outputs a $f(B) = B'$ such that B' is inside B but not intersecting the computably nowhere dense set.
  Example: Any line is computably nowhere dense.



- X is a *separator* of G and H if $X \subseteq G$ and $X \cap H = \emptyset$.

Bit and MRCA Computable: Alternate Characterization

Theorem

If $X \subseteq [0,1]^2$ is a set whose boundary is computably nowhere dense, then the following two conditions are equivalent.

(1) *X is (bit) computable.*

(2) *X is a separator of two computably open sets whose union is dense on $[0,1]^2$.*

Intuition: Being bit computable is the same as being able to cover the plane with accept and reject sets of rational balls.

Bit and MRCA Computable: Alternate Characterization

Why is the boundary condition needed?

$X = \left[0, \frac{1}{2}\right]^2 \cup (\mathbb{Q} \cap [0, 1])^2$

- X is computable.
- X is not even open!
  - Open and closed set.
  - No balls can cover $\mathbb{Q}^2$ anyways
- Boundary of $(\mathbb{Q} \cap [0, 1])^2$ is the unit square (not nowhere dense).

Bit and MRCA Computable: Main Theorem

Theorem

If $X \subseteq [0,1]^2$ is a set whose boundary is computably nowhere dense, then X is computable if and only if X is MRCA-computable.

Intuition: We can write an MRCA rule set to color the unit square according to X iff X is computable (given an X with a "thin" boundary).

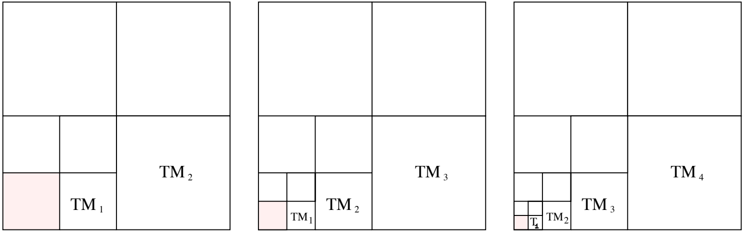
Bit and MRCA Computable: Main Proof Outline

Outline of Proof:

- If $X \subseteq \mathbb{R}^2$ is a computable set, then there exist computably open sets G and H for X as described earlier (union dense on $[0,1]^2$, etc.).
- Use algorithms for these sets in this construction: Take a ball as input, enumerate G and H, and output green (red) if the input ball is in G (H).
- Transform that algorithm into a one-dimensional cellular automata. Call this state and rule set a *computational unit* C_M.

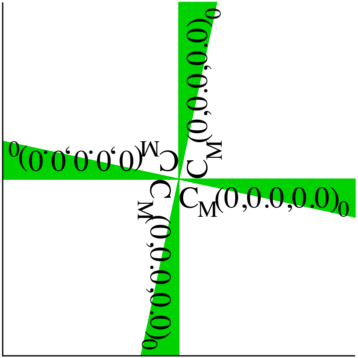
Bit and MRCA Computable: Main Proof Outline

- Computational units C_M are setup to use cells half their size to the left, double their size to the right.
  - Note: An MRCA cell can still read its neighbor to either side.



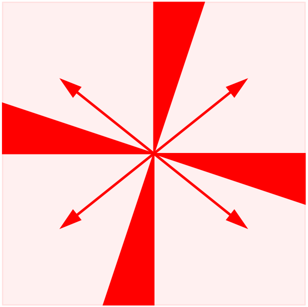
Bit and MRCA Computable: Main Proof Outline

- Add rules to color the space used by C_M when the input value is accepted.
- Rotate copies of that unit on a two-dimensional grid.



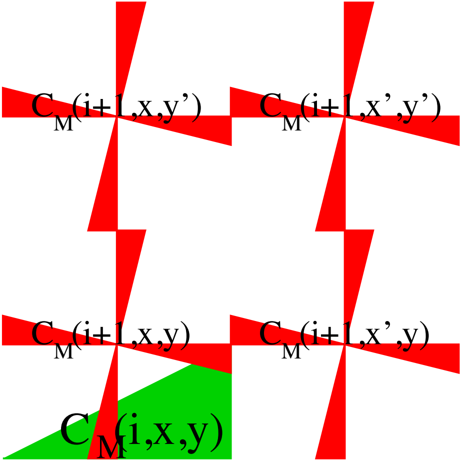
Bit and MRCA Computable: Main Proof Outline

- When any cell reads a colored cell to the (north or south) and (east or west), change to that color.



Bit and MRCA Computable: Main Proof Outline

- New addresses can be initialized by periodically pausing computation to create child computation units



Conclusions

- Briefly examined work by Brattka and Weihrauch [1], Braverman [2], and others to create a solid basis for computing sets of real numbers.
- Defined the MRCA model as a modification of 2D CA.
- Related MRCA computation to bit computation.
- Showed that many bit computable sets of real numbers are MRCA computable.

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Questions?