The partial orderings of the c.e. ibT- and cl-degrees are not elementarily equivalent

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When Turing reducibility is not enough

- Question: Given any (noncomputable) set $A \subseteq \mathbb{N}$, how do the sets $A = A(0)A(1)\ldots$ and $2A = A(0)0A(1)0A(2)0\ldots$ compare? Are they equally "difficult"?

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- If "equally difficult" means that the sets are Turing-computable from each other as an oracle with computably bounded use functions (*wtt-reducible*), the answer is still Yes, since the use functions of both reductions are bounded by $f(n) = 2n$.

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- If "equally difficult" means that the sets are Turing-computable from each other as an oracle with computably bounded use functions (*wtt-reducible*), the answer is still Yes, since the use functions of both reductions are bounded by $f(n) = 2n$.
- But: Even if A does not contain an infinite computable subsequence, $2A$ does. So intuitively, $2A$ should be easier than A.
- Is there a reducibility capturing this intuition?

ibT- and cl-reducibility

Definition

Let $A, B \subseteq \mathbb{N}$ be sets. Then A is *computable Lipschitz-(cl-)reducible to* B if there is a Turing reduction of A to B which to decide whether $x \in A$ only asks oracle questions "$y \in B$?" with $y \leq x + c$ for some constant c (independent of x).

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Remark:

This constraint on the use function of the reduction is very strong. In fact, if we would require the oracle questions asked for deciding whether $x \in A$ to be *strictly* smaller than x, then we would not get a reflexive reducibility notion anymore.

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Note that this shows that the three reducibilites $\leq_{wtt}$, $\leq_{cl}$ and $\leq_{ibT}$ differ already on the computably enumerable (c.e.) sets.

The degree structures of the strong reducibilities

Now look at the three degree structures $\mathcal{D}_{wtt}$, $\mathcal{D}_{cl}$ and $\mathcal{D}_{ibT}$, consisting of the equivalence classes of c.e. sets induced by *wtt*-, *cl*- and *ibT*-equivalence, respectively, with the induced partial orderings $\deg_r(A) \leq deg_r(B) \Leftrightarrow A \leq_r B$ for $r \in \{wtt, cl, ibT\}$.

Questions:

1. Is there a sentence in the language of first-order predicate logic with $<$ as the only additional relation symbol which is satisfied by $\mathcal{D}_{wtt}$ but not by $\mathcal{D}_{cl}$?

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1. Is there a sentence in the language of first-order predicate logic with $<$ as the only additional relation symbol which is satisfied by $\mathcal{D}_{wtt}$ but not by $\mathcal{D}_{cl}$?
2. Is there such a sentence which is satisfied by $\mathcal{D}_{cl}$ but not by $\mathcal{D}_{ibT}$?

Answer to Question 1

Yes! Two examples:

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   Because:

   - $deg_{wtt}(A \oplus B)$ is the supremum of $deg_{wtt}(A)$ and $deg_{wtt}(B)$.
   - But there are pairs of c.e. *cl*-degrees which do not have a least upper bound.

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We would like to try the sentence $\forall x(x = x + 1)$, where for $x = deg(A)$, $x + 1 = deg(A + 1)$. But there is no obvious way to express this in first order predicate logic with $\{<\}$!

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- Up to now, all attempts to embed specific finite lattices into the c.e. *ibT*-degrees were successful. (Conjecture: All finite lattices are embeddable.)
  The methods could always be adapted to obtain embeddings into the c.e. *cl*-degrees.

Cuppable degrees

Definition

Let $(P, \leq)$ be a partial ordering and let $a \in P$. An element b of P is *a-noncuppable* if $b \leq a$ and for all $c < a$, a is not the least upper bound of b and c (i.e. $b \cup c$ does not exist or $b \cup c < a$).

Noncuppable degrees in the ibT-degrees

Theorem

In the degree structure of $\mathcal{D}_{ibT}$, for any $\mathbf{a} > \mathbf{0}$, the $\mathbf{a}$-noncuppable degrees are bounded above by $\mathbf{a} + \mathbf{1}$.

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Proof idea.

Let $\mathbf{0} < \mathbf{a}$ and let $\mathbf{b} \leq \mathbf{a}$.

- Let A and B be c.e. sets such that $deg_{ibT}(A) = \mathbf{a}$ and $deg_{ibT}(B) = \mathbf{b}$ and such that there are computable 1-1-enumerations $(a_s)_{s \geq 0}$ and $(b_s)_{s \geq 0}$ of A and B with $a_s \leq b_s$ for all s ("B must permit A").

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- It holds that $deg_{ibT}(A) = deg_{ibT}(B) \cup deg_{ibT}(A_1)$, where $A_1 = \{a_s : a_s < b_s\}$.

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- It holds that $deg_{ibT}(A) = deg_{ibT}(B) \cup deg_{ibT}(A_1)$, where $A_1 = \{a_s : a_s < b_s\}$.
- If $B \not\leq_{ibT} A + 1$, then $A_1 <_{ibT} A$. Hence $deg_{ibT}(B)$ is $\mathbf{a}$-cuppable.

□

Noncuppable degrees in the cl-degrees

Definition

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Theorem

Let $\mathbf{a}$ *and* $\mathbf{b}$ *be c.e. cl-degrees such that* $\mathbf{a}$ *is 2-scattered and* $\mathbf{b} < \mathbf{a}$*. Then there is an* $\mathbf{a}$*-noncuppable cl-degree* $\mathbf{c} \leq \mathbf{a}$ *such that* $\mathbf{c} \not\leq \mathbf{b}$*. Hence, the* $\mathbf{a}$*-noncuppable cl-degrees have no upper bound less than* $\mathbf{a}$*.*

Proof idea

Let A be c.e. and 2-scattered and let $B <_{cl} A$ be a c.e. set. We want to construct a c.e. set $C \leq_{cl} A$ whose degree does not cup to $deg_{cl}(A)$ and such that $C \not\leq_{cl} B$.

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1. By replacing B by some shift $B + k$, assume $B \leq_{ibT} A$. W.l.o.g. assume there are computable 1-1-enumerations $(a_s)_{s \geq 0}$ and $(b_s)_{s \geq 0}$ of A and B with $a_s \leq b_s$ for all s.

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2. Construct a c.e. set C with enumeration $(c_s)_{s \in \mathbb{N}}$ that satisfies the following:
   - $\forall s(a_s \leq c_s < 2a_s)$
   - $\forall s(c_s$ is even$)$
   - $\forall e \exists s_e \forall s \geq s_e(c_s > a_s + e)$
   - $R_e : C \not\leq_{cl} B$ via the e-th cl-reduction

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   The construction is a straightforward finite-injury priority argument. $C \leq_{cl} A$ and $C \not\leq_{cl} B$ are obvious.

Proof idea (continued)

3. Show that C does not cup to A: Assume otherwise, i.e. $deg_{cl}(A) = deg_{cl}(C) \cup deg_{cl}(D)$ for some c.e. $D <_{cl} A$. We can show that then D can be assumed to contain only odd numbers. Then D is disjoint from C, implying $A \leq_{cl} C \cup D$. Let e be the constant overhead of the use of this cl-reduction.

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   But then $A \leq_{ibT} D$: Assume that $A \restriction x$ has already been computed. To compute whether $x = 2^m \in A$, using oracle D only, answer oracle queries "$y \in C \cup D$?" for $y \leq x$ as follows.

   - If y is odd, then use the oracle D to decide whether $y \in C \cup D$ (since C contains only even numbers).
   - If y is even and $y \geq x$, then $y \leq x + e$. But for almost all m, $C \cap [2^m, 2^m + e] = \emptyset$. So answer negatively.
   - If y is even and $y < x$, then find $m' < m$ with $y \in [2^{m'}, 2^{m'+1})$. If $2^{m'} \notin A$, then $y \notin C$; otherwise find the unique s such that $a_s = 2^{m'}$, then $y \in C$ iff $y = c_s$.

Answer to Question 2!

Corollary

The Σ^0_5-theories of the c.e. ibT$-$ and the c.e. cl$-$degrees are different.

Proof.

Let

$$\sigma = \exists a \forall b (b < a \to \exists c (c \not\leq b \wedge \forall d < a \exists e (d \leq e \wedge c \leq e \wedge a \not\leq e))).$$

Then $\mathcal{D}_{cl} \models \sigma$, but $\mathcal{D}_{ibT} \not\models \sigma$. □

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Open Question: We know that the Σ^0_1-theories of $\mathcal{D}_{ibT}$ and $\mathcal{D}_{cl}$ are equal.
What about the Σ^0_2, Σ^0_3 and Σ^0_4 theories?

Some further results on cuppable degrees

- Ambos-Spies has shown that, for any c.e. noncomputable set $A \subseteq 2\mathbb{N}$, $deg_{ibT}(A+1)$ is the *greatest* c.e. *ibT*-degree that does not cup to $deg_{ibT}(A)$.
- In general, this is not true: For every nontrivial computable shift f there is a noncomputable c.e. set A such that $deg_{ibT}(A_f)$ cups to $deg_{ibT}(A)$.
- Open Question: Is there for every noncomputable c.e. set A an unbounded shift f such that $deg_{ibT}(A_f)$ does not cup to $deg_{ibT}(A)$?

Thank you for your attention!