Incomputability of Simply Connected Planar Continua

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Introduction

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- The main theme of this talk is *Computability Theory for Connected Spaces*.

Definition

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1. $x \in \mathbb{R}^n$ is *computable* if $\{e \in \mathbb{N} : x \in B_e\}$ is c.e.
   Equivalently, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is computable iff x_i is computable for each $i \leq n$.

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3. (*Measure*) Not every nonempty *positive measure* Π^0_1 set in $\mathbb{R}^1$ contains a computable point.
4. (*Connectedness*) What about *connected*, *simply connected*, or *contractible* Π^0_1 sets?

Connected Π^0_1 Sets

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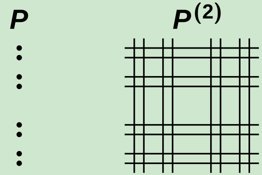
Fact

1. There exists a nonempty *connected* Π^0_1 subset $P^{(2)} \subseteq \mathbb{R}^2$ without computable points.
2. There exists a nonempty *simply connected* Π^0_1 subset $P^{(3)} \subseteq \mathbb{R}^3$ without computable points.

Example



$P^{(n)} = \bigcup_{k<n}([0,1]^k \times P \times [0,1]^{n-k-1})$ for $P \subseteq [0,1]$.

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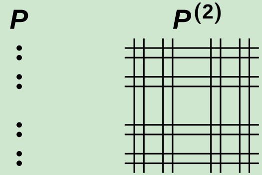

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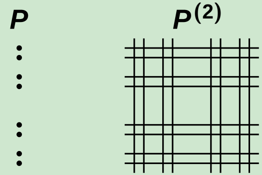

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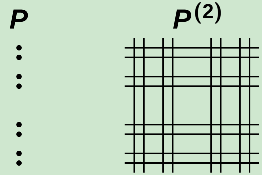

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- $P^{(2)} \subseteq [0,1]^2$ is a connected Π^0_1 set without computable points.
- $P^{(3)} \subseteq [0,1]^3$ is a simply connected Π^0_1 set without computable points.

- X is n-connected $\iff$ the first $n+1$ homotopy groups vanish identically.
- X is path-connected $\iff$ X is 0-connected.
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- X is contractible $\iff$ the identity map on X is null-homotopic.
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- Let $P \subseteq [0,1]$ be a Π^0_1 set without computable points.
- $P^{(n+2)} \subseteq [0,1]^{n+2}$ is n-connected, but not $n+1$-connected.

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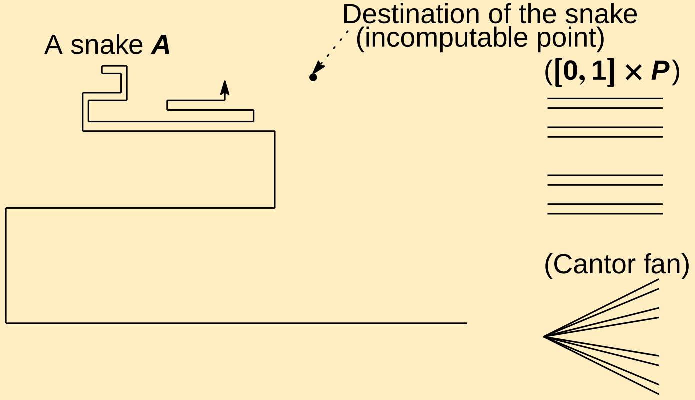
- All path-component of $[0,1] \times P$ will be bundled at the destination end-point.
- Thus, the desired Π^0_1 set will be homeomorphic to the Cantor fan.

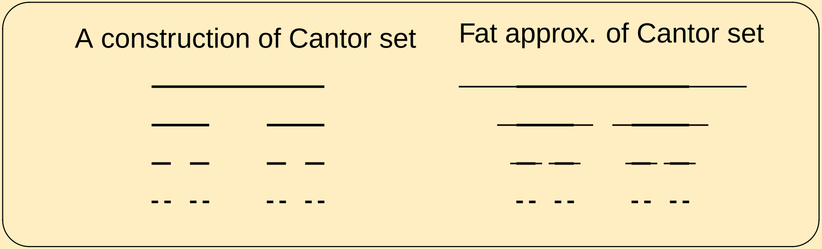
Proof Idea

Stretch $[0,1] \times P$ along a *stray snake* A.



The desired Π^0_1 set D will be homeomorphic to the Cantor fan.

A fat approximation of Cantor set:



- P: a Π^0_1 subset of Cantor set.
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- l_s, r_s: the leftmost and rightmost of P_s.

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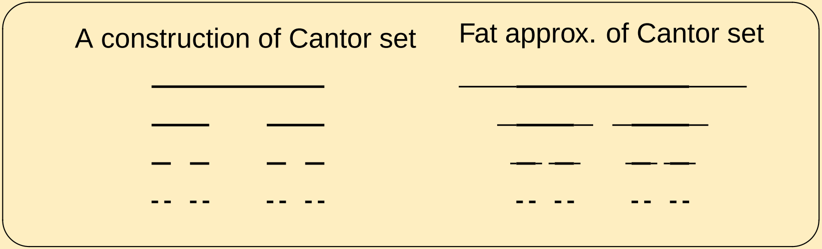

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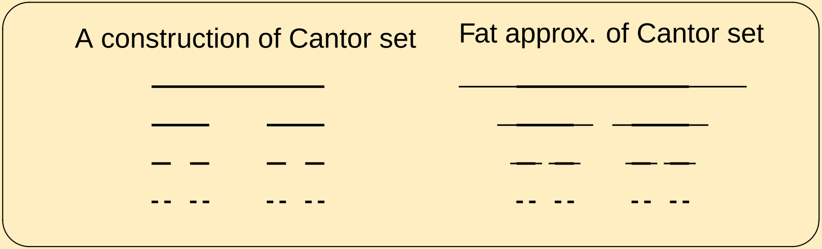

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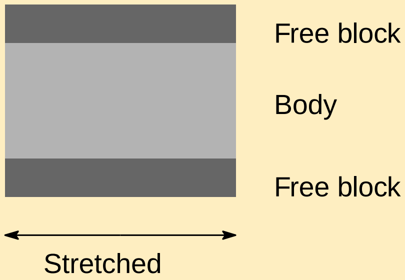
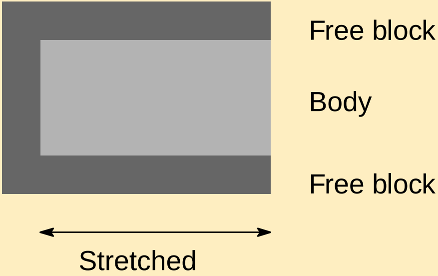
- We call these intervals $I^l_s, I^r_s \subseteq P_s \setminus P_{s+1}$ free blocks.

Prepare a stretched Π^0_1 class $D^-_0 = P \times [0,1]$.



- $P \subseteq \mathbb{R}^1$: a Π^0_1 set without computable points.
- P_s: a fat approximation of P (Note that $P = \bigcap_s P_s$).
- $D^-_0 = [0,1] \times P_0$.

D_0 is the following connected closed set.

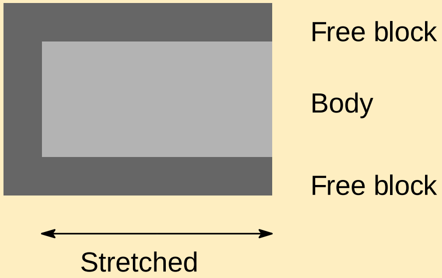


The desired Π^0_1 set D will be obtained by carving D_0.

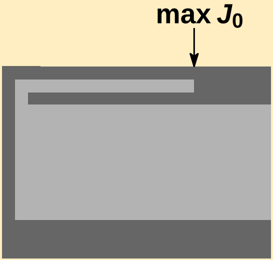
Destination

- $\alpha \in \mathbb{R}$: an incomputable left-c.e. real.
- There is a computable sequence $\{J_s\}$ of rational open intervals s.t.
  - $\min J_s \to \alpha$ as $s \to \infty$.
  - $\mathrm{diam}(J_s) \to 0$ as $s \to \infty$.
  - Either $J_{s+1} \subset J_s$ or $\max J_s < \min J_{s+1}$, for each s.

Our construction starts with D_0.



By carving free blocks, stretch P_0 toward $\max J_0$.



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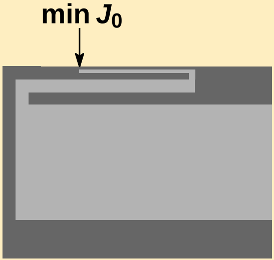
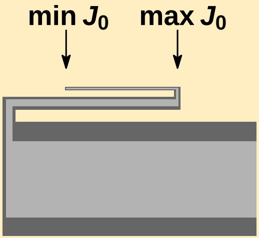
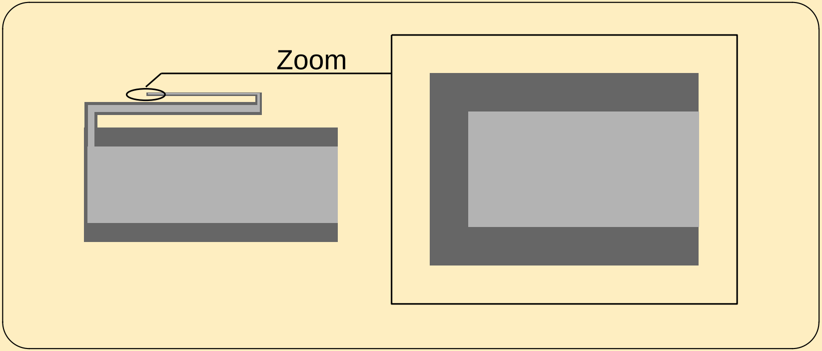


Proceed one step with a fat approximation of P.



D_1 is defined by this,



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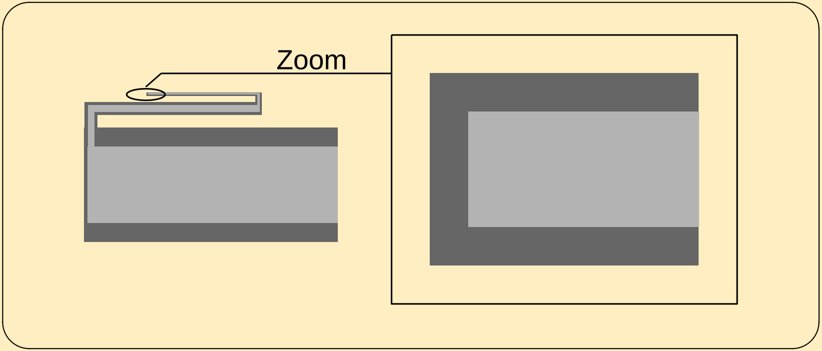

- If $J_1 \subset J_0$, then the construction of D_2 is similar as that of D_1.
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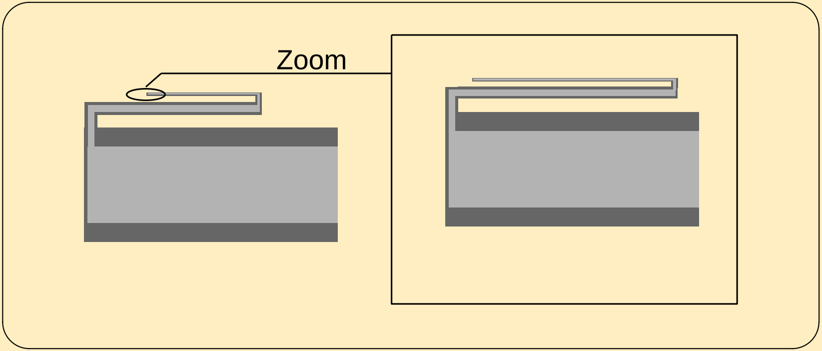

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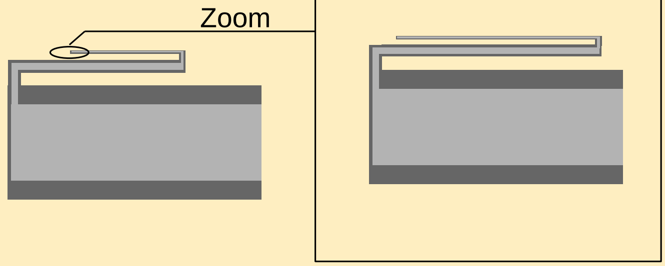

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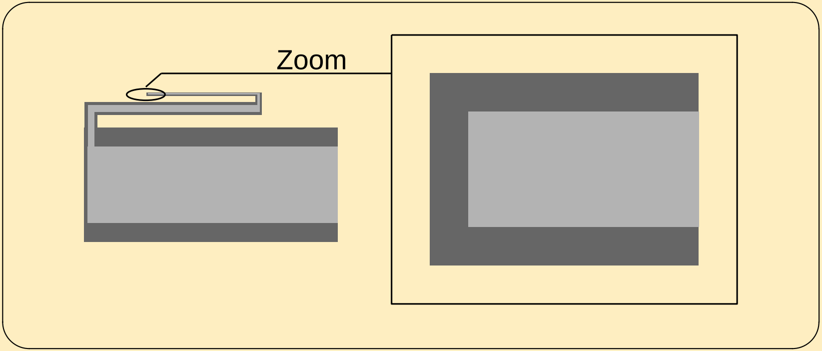

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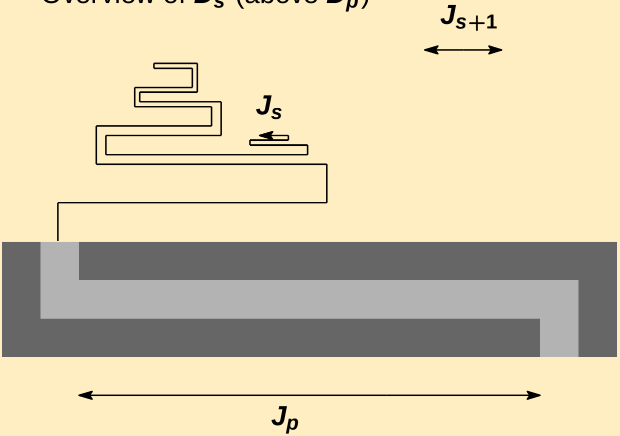
- In general, similar for $J_{s+1} \subset J_s$.
- Only the problem is the case of $J_{s+1} \not\subset J_s$!

In the case of $J_{s+1} \not\subset J_s$:

Overview of D_s (above D_p)



Pick the greatest $p \leq s$ such that $J_{s+1} \subset J_p$.

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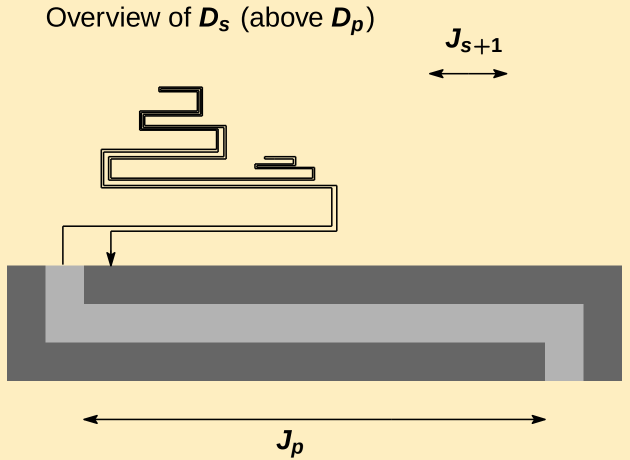

Go back to D_p by caving free blocks into the shape of P.

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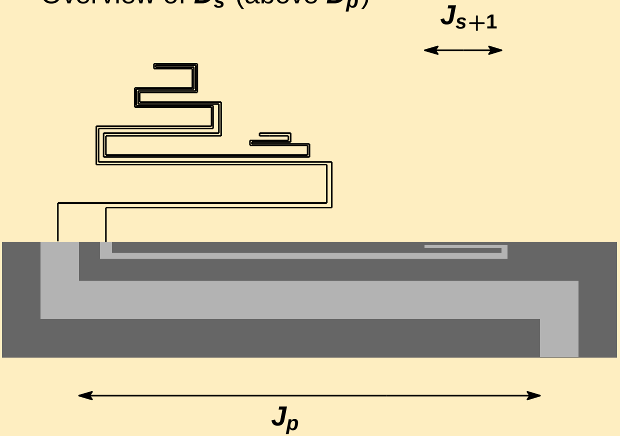

By caving free blocks on D_p into the shape of P, stretch toward **max** J_{s+1} and back to **min** J_{s+1}.

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- Therefore, D is homeomorphic to Cantor fan, and contractible.
- Stretching $[0,1] \times P$ cannot introduce new computable points.

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- D is path-connected by the property of an approximation $\{J_s\}$ of the incomputable left-c.e. real α.
- Therefore, D is homeomorphic to Cantor fan, and contractible.
- Stretching $[0,1] \times P$ cannot introduce new computable points.
- Of course, (α, y) is also incomputable.

Main Theorem (Restated)

Not every nonempty contractible planar Π^0_1 set contains a computable point.

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- Therefore, D is homeomorphic to Cantor fan, and contractible.
- Stretching $[0,1] \times P$ cannot introduce new computable points.
- Of course, (α, y) is also incomputable.
- Hence, D has no computable points.

Corollary

For every Π^0_1 class P, there is a contractible planar Π^0_1 set D such that D is Turing-degree-isomorphic to P.

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For every Π^0_1 class P, there is a contractible planar Π^0_1 set D such that D is Turing-degree-isomorphic to P.

Definition (RCA$_0$)

A sequence $(B_i)_{i\in\mathbb{N}}$ of open rational balls is *disk-like* if $\bigcup_{i<n} B_i$ is homeomorphic to $(0,1)^2$ for any $n \in \mathbb{N}$.

Corollary

The following are equivalent over **RCA$_0$**:

- **WKL$_0$**: Every infinite tree has a path;
- **Heine-Borel**: Every covering of $[0,1]$ has a finite subcovering.
- **Heine-Borel(Disk)**: Every disk-like covering of $[0,1]^2$ has a finite subcovering.

Question

- What about Medvedev degrees of contractible planar Π^0_1 sets?
- Does every nonempty locally connected planar Π^0_1 set contain a computable point?

Thank you!