Mass Problems and Relative Learnability

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Motivation

- Medvedev (1955) introduced the Medvedev degrees to formulating the semantics of intuitionistic prop. calculus.
- The supermum $\otimes$ can be viewed as the conjunction (AND).
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  - *a learnable disjunction with mind-change-bound* α,
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- MAIN THEME:
  Interaction of *learnable disjunction* and *relative learnability*!

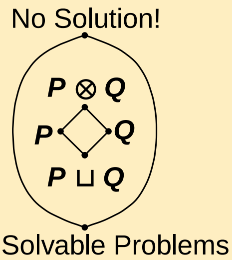
- A mass problem is a subset of ω^ω.
- We think of each mass problem as the solution set of some mathematical problem.

Definition (Medvedev, 1955)

Let P, Q be mass problems (i.e., $P, Q \subseteq \omega^\omega$).
$P \leq_M Q$ if there is a computable map $\Phi_e : Q \to P$, i.e.,
$P \leq_M Q \iff (\exists e)(\forall g \in Q)\ \Phi_e(g) \in P$.

$P \leq_M Q$ (via Φ_e): If we have a solution g to Q then we can calculate a solution to P by executing the algorithm Φ_e with an oracle g.

The Medvedev Lattice

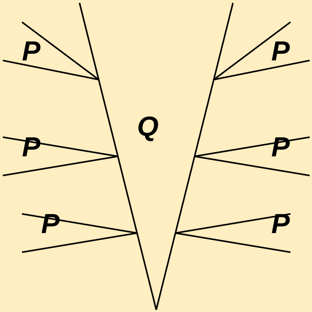


- (top) $\emptyset$.
- (bottom) Any $P \subseteq \omega^\omega$ containing a computable element.
- (sup) $P \otimes Q = \{f \oplus g : f \in P \ \& \ g \in Q\}$.
- (inf) $P \sqcup Q = \{0f : f \in P\} \cup \{1g : g \in Q\}$.

Definition

$P \subseteq 2^\omega$ is a Π^0_1 class if it is the set of all paths of a computable tree $T_P \subseteq 2^{<\omega}$. Equivalently, it is Π^0_1 definable in 2^ω.

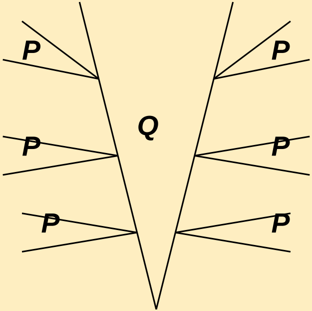
- Let $L(P)$ denote the set of all leaves of a tree T_P, i.e., $L(P) = \{\rho j : \rho \in T_P \ \& \ (\forall i)\ \rho i \notin T_P\}$.
- $L(P)$ is computable whenever T_P is computable.



Definition

For Π^0_1 classes $P, Q \subseteq 2^\omega$,

$$Q \nabla P = Q \cup \bigcup_{\rho \in L(Q)} \rho P.$$



$Q \triangledown P$ represents the mass problem:

- "First we try to solve Q",
- "If we failed to solve Q, then next we try to solve P".

i.e., Solve "P or Q" *with a mind-change*!

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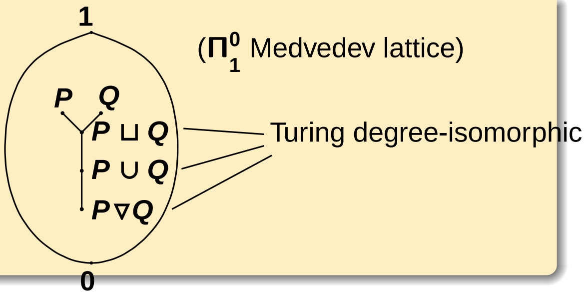
Basic Properties

For Π^0_1 classes $P, Q \subseteq 2^\omega$,

- $Q \nabla P$ is also Π^0_1.
- $Q \nabla P$ is Turing-degree-isomorphic to $P \cup Q$.

Theorem

- $P \triangledown Q \leq_M P \cup Q \leq_M P \sqcup Q$ for any Π^0_1 classes $P, Q \subseteq 2^\omega$.
- $P \triangledown Q <_M P \cup Q <_M P \sqcup Q$ for some Π^0_1 classes $P, Q \subseteq 2^\omega$.



We think of $\triangledown$ as a kind of *meet* (or *disjunction*) operator, such as $\sqcup$ and $\cup$.

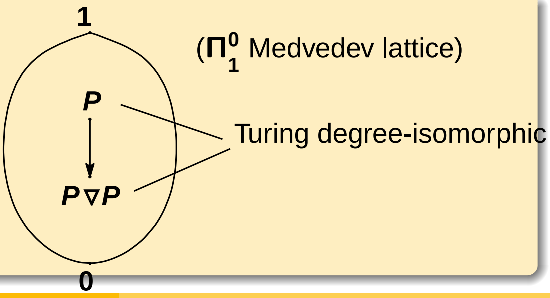
Theorem

For Π^0_1 classes $P, Q \subseteq 2^\omega$, $Q \triangledown P$ does not Medvedev-cup to P, i.e., $(\forall P, Q, R \in \Pi^0_1)\ (P \leq_M (Q \triangledown P) \otimes R \ \rightarrow\ P \leq_M R)$.

In particular, $Q \triangledown P <_M P$ for any special Π^0_1 class P, Q.

Corollary

Every nonzero Medvedev degree has the anti-cupping property, i.e., $\mathcal{P}_M \models (\forall a)(\exists b)(\forall c)\ (a \leq b \vee c \ \rightarrow\ a \leq c)$.

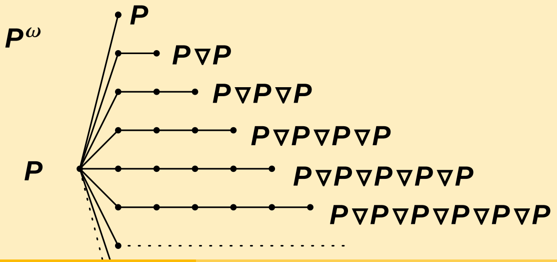


Corollary

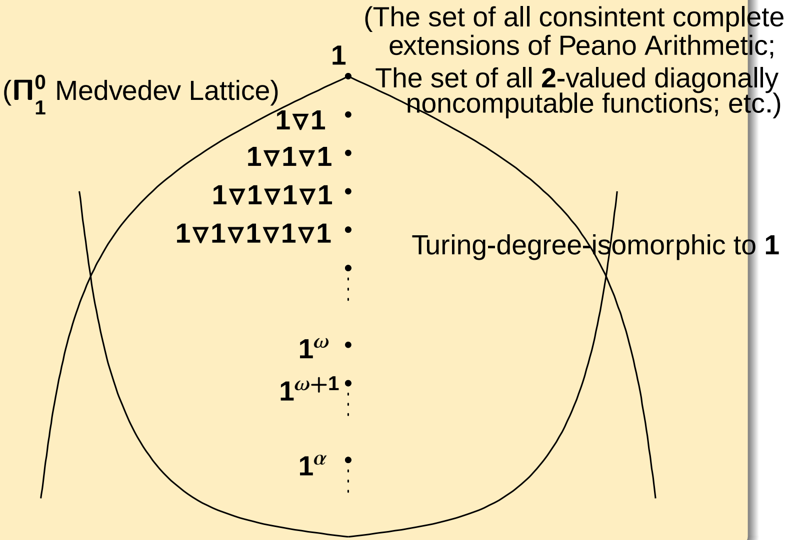
$(\forall$ special Π^0_1 class $P \subseteq 2^\omega)(\exists\ (P^\alpha)_{\alpha<\omega_1^{CK}})$ such that

- $P^0 = P$; P^α is Π^0_1 for any $\alpha < \omega_1^{CK}$.
- If $\alpha < \beta < \omega_1^{CK}$, then P^β does not Medvedev-cup to P^α.
- P^α is degree-isomorphic to P^β for any $\alpha, \beta < \omega_1^{CK}$.

We can think of P^α as the mass problem saying: "solve P *with mind-change-bound α*!"



Embedding of Hyperarithmetical Hierarchy



Definition (Gold 1965)

- Ψ is a learner if it is computable function from $\omega^{<\omega}$ to ω.
- A learner Ψ learns $\alpha \in \omega^\omega$ if $\Phi_{\lim_n \Psi(\alpha \upharpoonright n)} = \alpha$.

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Definition (Relative Learnability)

- $\Gamma : \omega^\omega \to \omega^\omega$ is learnable if
  $(\exists \Psi)(\forall \alpha \in \mathrm{dom}(\Gamma))\ \Phi_{\lim_n \Psi(\alpha \upharpoonright n)}(\alpha) = \Gamma(\alpha)$.
  Here Ψ ranges over all learners.
- $P \leq_l Q \iff$ there is a learnable map $\Gamma : Q \to P$.

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Proposition

- $\leq_l$ is preordering.
- $P \leq_M Q \to P \leq_l Q \to P \leq_w Q$ for any $P, Q \subseteq \omega^\omega$.
  (Here $P \leq_w Q \iff (\forall g \in Q)(\exists f \in P)\ f \leq_T g$.)
- $Q \nabla P \equiv_l Q \cup P \equiv_l Q \sqcup P$ for any Π^0_1 classes $P, Q \subseteq \omega^\omega$.

Definition

$$\nabla P = \bigcup_n (\underbrace{P \triangledown P \triangledown \ldots \triangledown P}_{n}).$$

Proposition

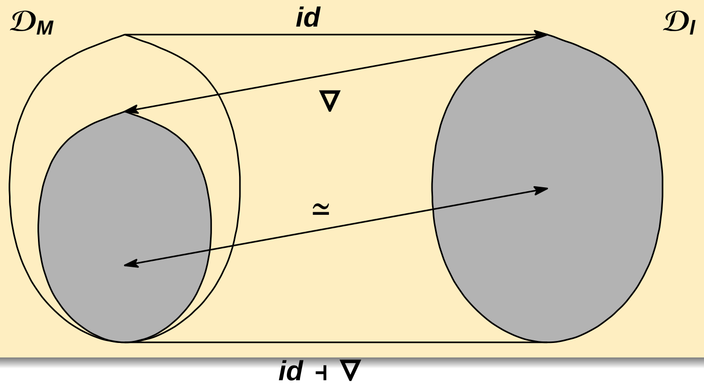
- ∇P is a Σ^0_2 class.
- $\nabla P \equiv_l P$.

Example

- Let **MLR** denote the set of all Martin-Löf random reals.
- $\nabla P = \mathbf{MLR}$ for any nonempty Π^0_1 class $P \subseteq \mathbf{MLR}$.
- For example, $\nabla(\{\alpha \in 2^\omega : (\forall n)\ K(\alpha \upharpoonright n) \geq n - 1\}) = \mathbf{MLR}$.

Theorem

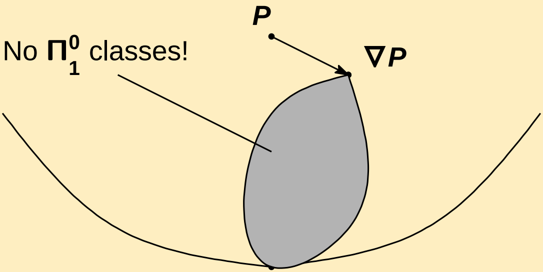
- $P \leq_I Q \iff \nabla P \leq_M Q \iff \nabla P \leq_M \nabla Q$.
- $\triangledown, \nabla$ are well-defined on the Medvedev lattice $\mathcal{D}_M$.
- $\mathcal{D}_I = \nabla \mathcal{D}_M = \{\nabla a : a \in \mathcal{D}_M\}$.



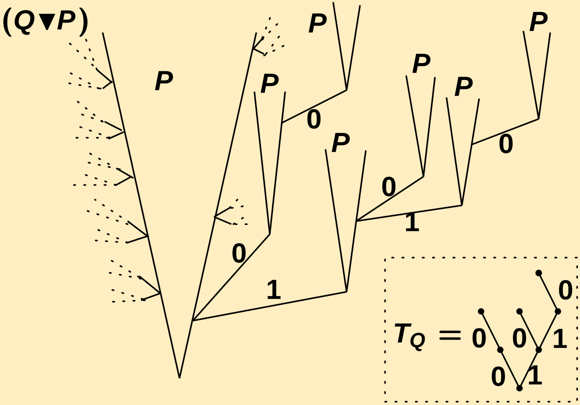
Theorem

For every special Π^0_1 classes $P, Q \subseteq 2^\omega$, we have $Q \not\leq_M \nabla P$.

More generally, for any finite partition $\{P_i\}_{i<k}$ of ∇P, there exists $i < k$ such that $Q \not\leq_M P_i$.

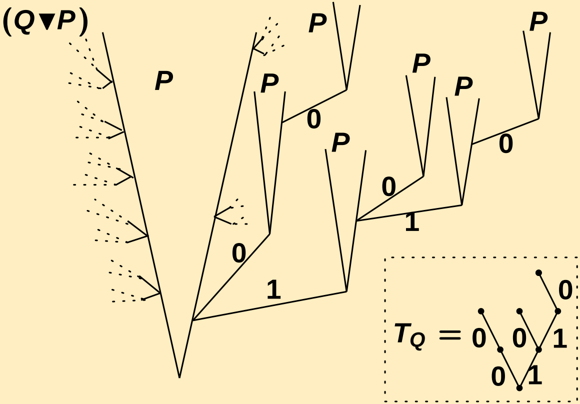


- Recall that P^α is obtained by iterating ∇ along a well-founded tree of ordinal α.
- The mass problem $Q \blacktriangledown P$ is obtained by iterating ∇ along a ill-founded tree T_Q.



Definition

$$Q \blacktriangledown P = \{\sigma_0 m_0 \sigma_1 m_1 \dots \sigma_i : (\forall j < i)\sigma_j \in L(P) \;\&\; \sigma_i \in T_P \;\&\; \langle m_0 \dots m_{i-1} \rangle \in T_Q\}.$$



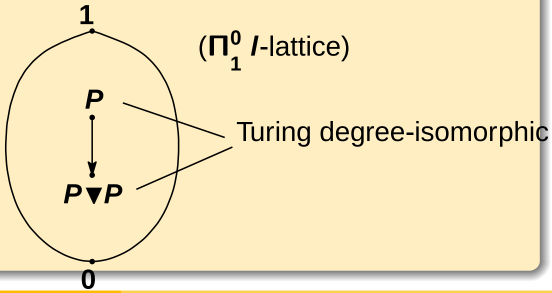
Theorem

For Π^0_1 classes $P, Q \subseteq 2^\omega$,

- $Q \blacktriangledown P$ does not I-cup to P,
  i.e., $(\forall P, Q, R \in \Pi^0_1)\ (P \leq_I (Q \blacktriangledown P) \otimes R \ \rightarrow\ P \leq_I R)$.

Corollary

Every nonzero I-degree has the anti-cupping property,
i.e., $\mathcal{P}_I \models (\forall a)(\exists b)(\forall c)\ (a \leq b \vee c \ \rightarrow\ a \leq c)$.



We have already defined

- an intuitionistic disjunction $\boldsymbol{P} \sqcup \boldsymbol{Q}$,
- a learnable (semi-intuitionistic) disjunction $\boldsymbol{P} \triangledown \boldsymbol{Q}$.

We can also define a classical disjunction $\boldsymbol{P} \curlyvee \boldsymbol{Q}$.

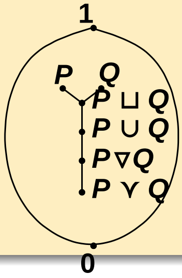
- $\boldsymbol{P} \curlyvee \boldsymbol{Q}$ represents the mass problem saying:
  Solve "$\boldsymbol{P}$ or $\boldsymbol{Q}$" *with a parallel procedure*!
- $\nabla \boldsymbol{P} \curlyvee \nabla \boldsymbol{Q}$ represents the mass problem saying:
  Solve "$\boldsymbol{P}$ or $\boldsymbol{Q}$" *with a team-learning procedure*!

Proposition

- $P \curlyvee Q \equiv_w P \sqcup Q$.
- $Q \blacktriangledown P \equiv_w Q \curlyvee \nabla P \leq_M Q \blacktriangledown P$.

Theorem

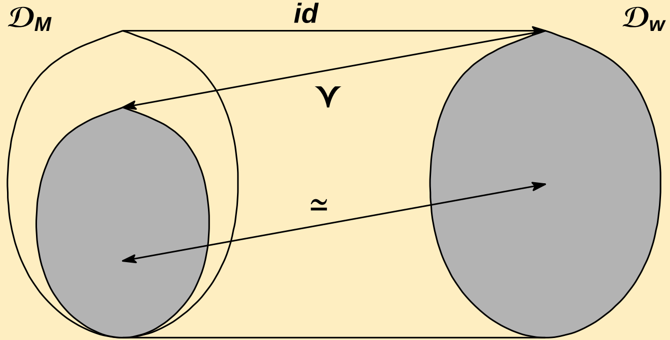
- $P \curlyvee Q \leq_M P \triangledown Q \leq_M P \cup Q \leq_M P \sqcup Q$.
- There exist Π^0_1 classes P and Q such that: $P \curlyvee Q <_I P \triangledown Q <_M P \cup Q <_M P \sqcup Q$.



(Π^0_1 Medvedev lattice)

Theorem

- $P \leq_w Q \iff \curlyvee P \leq_M Q \iff \curlyvee P \leq_M \curlyvee Q$,
  where $\curlyvee P = \bigcup_n (\underbrace{P \curlyvee P \curlyvee \dots \curlyvee P}_{n})$.
- $\curlyvee, \curlyvee$ are well-defined on the Medvedev lattice $\mathcal{D}_M$.
- $\mathcal{D}_w = \curlyvee \mathcal{D}_M = \{\curlyvee a : a \in \mathcal{D}_M\}$.

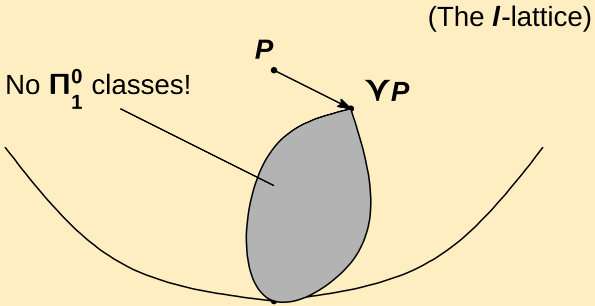


$id \dashv \curlyvee$

Theorem

For every special $\mathbf{\Pi}^0_1$ classes $P, Q \subseteq 2^\omega$, we have $Q \not\leq_I \curlyvee P$.

More generally, for any finite partition $\{P_i\}_{i<k}$ of $\curlyvee P$,
there exists $i < k$ such that $Q \not\leq_I P_i$.



Thank you!