Randomness and the Ergodic Decomposition

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Combination of Bernoulli measures

Combination of ergodic measures

Finite combination of ergodic measures

Tossing a random coin

Imagine the following process:

1. pick a *random* number $p \in [0, 1]$, according to some probability measure m over $[0, 1]$,
2. pick a *random* sequence $x \in \{0, 1\}^{\mathbb{N}}$ according to the coin-tossing (Bernoulli) measure B_p over $\{0, 1\}^{\mathbb{N}}$.

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This process can be described in one single step: let μ be the measure over $\{0, 1\}^{\mathbb{N}}$ defined by

$$\mu(A) = \int B_p(A)\, \mathrm{d}m(p).$$

The two-step process is equivalent to picking a *random* sequence according to μ.

Example: the Pólya urn

Consider an urn with **red** and **black** balls. At each round, pick a ball at random, put it back in the urn together with another ("fresh") ball of the same color.

Theorem (De Finetti)

The measure μ *over* $\{\mathbf{0}, \mathbf{1}\}$ *describing the process can be decomposed as*

$$\mu(A) = \int B_p(A)\,\mathrm{d}m(p)$$

for some measure m *over* $[0, 1]$.

Algorithmic randomness

- What if *random* is understood as *Martin-Löf random*?
- Are the two processes still equivalent?

Formally, the sets of sequence produces by the two processes are:

two-step process	one-step-process
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Reminder

x is μ-Martin-Löf random if $\exists c, \forall n, K(x\restriction_n) \geq -\log \mu[x\restriction_n] - c$.

The minimal such $c \in \mathbb{N}$ is denoted the ***randomness deficiency of*** x, denoted $d_\mu(x)$.

$$x \in \mathscr{R}_\mu \iff d_\mu(x) < \infty.$$

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Moreover,

Theorem (Davie/Kjos-Hanssen/H.)

p can be uniformly computed from x and a bound on $d_\mu(x)$.

Bernoulli measures are a particular class of ergodic measures.

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Let $\mathscr{S}$ be the set of stationary probability measures.

- $\mathscr{S}$ is a convex set: if $\nu_1, \nu_2 \in \mathscr{S}$ then $\mu = \alpha\nu_1 + (1-\alpha)\nu_2 \in \mathscr{S}$.

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- We call m a ***decomposition*** of μ.
- $\mu \in \mathscr{S}$ is called ***ergodic*** if it has no non-trivial decomposition (it is an extreme point of $\mathscr{S}$).

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Theorem (Ergodic decomposition, Choquet)

Every stationary measure μ has a unique decomposition m supported on the ergodic measures, i.e. satisfying $m(\mathscr{E}) = 1$.

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Hence picking a *random* sequence x according to μ is equivalent to:

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Question

What if *random* is understood as ***Martin-Löf random***?

This time,

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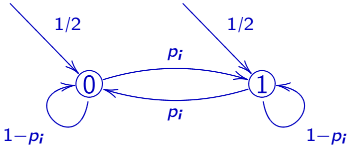
A counter-example (V'yugin, 1997)

- Let $p_i = 2^{-t_i}$ where t_i is the halting time of Turing machine M_i ($p_i = 0$ when M_i does not halt).
- Let μ_i be the measure over $\{0,1\}^{\mathbb{N}}$ corresponding to the Markov chain:



- The combination $\mu = \sum_i 2^{-i}\mu_i$ is computable, stationary, but its decomposition m is ***not*** computable.

When m is computable

m is computable $\Rightarrow$ μ is computable, but the converse is false.

Theorem

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Moreover, if $x \in \mathscr{R}_\mu$ *then:*

- *among* $\mathscr{R}_{\boldsymbol{m}}$*, only one* $\nu = \nu_x$ *makes* x *random,*
- ν_x *is ergodic,*
- ν_x *can be recovered from* x *as*

$$\nu_x[w] = \lim \mathit{freq}(w \in x).$$

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$$\nu_x[w] = \lim \mathit{freq}(w \in x).$$

Question

Can ν_x be *effectively* recovered from x?

When m is computable

Theorem (H.)

m is computable

$$\Longrightarrow$$

the function $x \mapsto \nu_x$ is μ-layerwise computable, i.e. ν_x can be uniformly computed from $x \in \mathscr{R}_\mu$ and a bound on $d_\mu(x)$.

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Let $\mathscr{C}$ be an effective compact class (i.e. Π_1^0) of ergodic measures.

Proposition

For every measure m supported on $\mathscr{C}$, the combination μ is computable from m.

Example

The class $\mathscr{C} = \mathsf{Ber}$ of Bernoulli measures is effectively compact.

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Assume the decomposition of μ is finite[1]:

$$\mu = \alpha_1 \nu_1 + \ldots + \alpha_p \nu_p.$$

[1] $\alpha_i > 0$, $\alpha_1 + \ldots + \alpha_p = 1$, $\nu_i \neq \nu_j$, ν_i ergodic.

Finite combination of ergodic measures

Assume the decomposition of μ is finite[1]:

$$\mu = \alpha_1 \nu_1 + \ldots + \alpha_p \nu_p.$$

Theorem

If μ is computable then for every μ-random x, $\nu_x \in \{\nu_1, \ldots, \nu_p\}$ is ergodic.

Open questions

In the finite case,

- μ computable $\Rightarrow$ α_i, ν_i computable? (i.e., m computable?)
- $x \in \mathscr{R}_\mu \Rightarrow x \in \mathscr{R}_{\nu_x}$?

[1] $\alpha_i > 0$, $\alpha_1 + \ldots + \alpha_p = 1$, $\nu_i \neq \nu_j$, ν_i ergodic.

Open questions

- If μ is computable and $x \in \mathscr{R}_\mu$, is ν_x ergodic? $\mathscr{R}_\mu = \bigcup_{\nu \in \mathscr{R}_{\boldsymbol{m}}} \mathscr{R}_\nu$?
- Is there a ***finitely*** decomposable stationary measure, whose decomposition is not computable?
- Let $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ with μ_1, μ_2 ergodic and $\mu_1 \neq \mu_2$. If μ is computable, are μ_1, μ_2 computable?