Automorphisms of Computable Linear Orders and The Ershov Hierarchy.

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Outline

Section Guide

Introduction.

Question

Given an order type σ and a computable linear order of order type σ, what is the exact level in the arithmetical hierarchy extended via the Ershov hierarchy at which nonrigidity breaks down.

Theorem (Kierstead 1987)

For $\sigma \in \{2 \cdot \eta, \omega + \zeta\}$ there exists a computable linear order of order type σ which is Π^0_1 rigid—i.e. has no nontrivial automorphism. However every computable linear order of order type σ has a nontrivial Δ^0_2 automorphism.

Section Guide

ω-c.e. Sets.

Lemma (Schoenfield)

A set $A \subseteq \omega$ is Δ^0_2 iff there is a computable function $h : \omega \times \omega \to \{0, 1\}$, written $h_s(n)$, such that, for all $n \in \omega$,

(1) $h_0(n) = 0$,

(2) $lim_{s\to\infty} h_s(n) = A(x)$.

Definition

If A satisfies conditions (1) and (2) as also, for all $n \in \omega$

(3) $|\{\, s \mid h_{s+1}(n) \neq h_s(n) \,\}| \leq g(x)$

for some computable function $g : \omega \to \omega$, then A is said to be ω-c.e.

Exact Uniform Δ^0_2 Classes.

Definition

A class $\mathcal{F}$ of (partial) functions is said to be exact uniform Δ^0_2 if there exists a computable function $f : \omega \times \omega \times \omega \to \omega$, written $f_{e,s}(n)$, such that, for all $e, n \in \omega$, either

(1) $\lim_{s\to\infty} f_{e,s}(n)$ exists, or

(2) $\liminf_{s\to\infty} f_{e,s}(n) = \infty$

and, letting f_e denote the function approximated accordingly, i.e. $f_e(n) = \lim_{s\to\infty} f_{e,s}(n)$ if the latter exists and is undefined otherwise, $\mathcal{F} = \{f_e\}_{e\in\omega}$.

We call $\{f_{e,s}\}_{e,s\in\omega}$ an exact uniform Δ^0_2 approximation of $\mathcal{F}$.

We call any class $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ uniform Δ^0_2.

The ω-c.e. Functions.

Notation

Given an arithmetical predicate P, such as Π^0_1 or ω-c.e., a function $f : \omega \to \omega$ is said to be *P* if the graph of f is a P set.

Lemma

The class of ω-c.e. functions $\mathcal{F}_\omega$ is Δ^0_2 uniform.

Section Guide

The Main Theorem

Theorem

For any uniform Δ^0_2 class $\mathcal{F}$ there exists a computable linear order $\mathcal{L}$ of order type $2 \cdot \eta$ which is $\mathcal{F}$-rigid.

Background to the Proof.

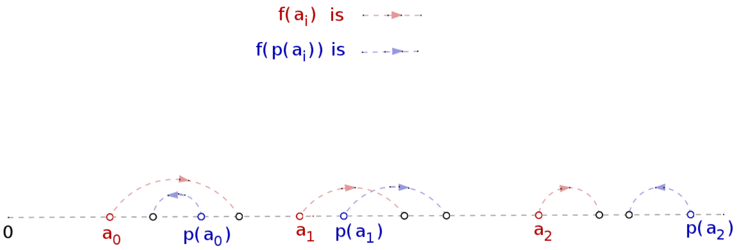
The Upwards Search Lemma

If $\mathcal{L} = \langle L, <_L \rangle$ is a linear order of order type $2 \cdot \eta$, $L = \omega$, $p : L \to L$ is the associated pairing function, and f is a non trivial automorphism of $\mathcal{L}$, then the set

$$K_f = \{ a \mid a \in L \ \& \ p(a) > a \ \& \ f(a) > a \ \& \ f(p(a)) > a \} . \tag{1}$$

is infinite (where $<$ is the usual ordering over ω).

Background to the Proof.



Special pairs $\{a_i, p(a_i)\}$ for f in $L=\omega$, seen under the usual ordering $0<1<2\ldots$

Background to the Proof.

- We construct $\mathcal{L} = \langle L, <_L \rangle$ with associated pairing function p so that $L = \omega$. At each stage s, we define finite approximations to L, $<_L$ and p. L_s is defined to be an initial segment of ω such that $L_s \subset L_{s+1}$, and both $<_L^s$ and p_s are defined with domain L_s.

- Consider $\mathcal{F}$, the uniform Δ_2^0 class of functions over ω stated in the theorem. Then, by definition, there exists an exact uniform Δ_2^0 class enumeration $\widehat{\mathcal{F}} = \{f_e\}_{e\in\omega}$ with associated uniform Δ_2^0 approximation $\{f_{e,s}\}_{e,s\in\omega}$, such that $\mathcal{F} \subseteq \widehat{\mathcal{F}}$. We pick $\widehat{\mathcal{F}}$, with its associated exact uniform Δ_2^0 approximation for use during the construction.

The Requirements.

The construction aims to satisfy for all e, the following requirements:

$$Q_e \quad : \quad p(e)\downarrow,$$
$$R_e \quad : \quad f_e \text{ is not a nontrivial automorphism of } \mathcal{L},$$

also the structural requirements that $P(e) = \{e, p(e)\}$ is an adjacent pair under $<_L$, that $\mathcal{L}$ is a dense linear order of these pairs (densification), and moreover the complexity requirment that $\mathcal{L}$ is computable.

Densification.

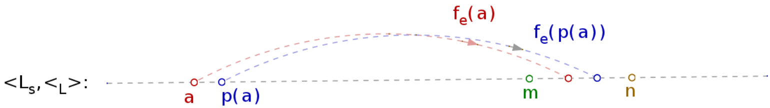
At every stage $s + 1$ there will be a finite amount of re-pairing activity carried out on behalf of the R requirements. At the end of the stage densify with new pairs.

The pairs already in L_{s+1} following re-pairing activity by R strategies: ∘ ∘

The pairs added to L_{s+1} the end of stage s+1: ∘ ∘

The densification of $<L_{s+1},<_L>$ uses the least numbers not yet used in L_{s+1}.

Diagonalising f_e at stage $s+1$.



Diagonalise against f by re-pairing $f_e(a)$ with m and $f_e(p(a))$ with n.

Note: m and n are the least numbers in ω-L_s.

Cohesion of the Q and R Strategies.

The re-pairing activity of the R strategies (aimed at defeating nontrivial automorphisms) conflicts with the Q strategies (aimed at ensuring that $p(n)$ converges on all $n \in \omega$).

However, using the upwards search lemma and the fact that $\{f_{e,s}\}_{e,s\in\omega}$ is an exact uniform Δ_2^0 approximation we can ensure that the re-pairing activity of the R strategies tends to ∞ (over the natural numbers).

Basic Idea of the Proof: satisfying R_e.

- At stage $s+1$ watch the activity of f_e over L_s, i.e. is $f_{e,s+1}(n) \neq f_{e,s}(n)$ for some $n \in L_s$ etc. (Remember that L_s is an initial segment of ω.)
- Define a list of numbers $L(e, s+1)$ such that, for each $n \in L(e, s+1)$, f_e appears to map n to some $m > n$, and such that there f_e appears stable on all numbers $p < n$.
- Define the set $E(e, s+1)$ relative to $L(e, s+1)$ and restrain $E(e, s+1)$ against re-pairing by lower priority R strategies.

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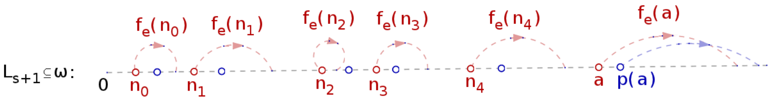
Basic Idea of the Proof: satisfying R_e.

The R_e strategy works under the following assumptions.

Assumption 1. There exists a finite set $\widehat{E}(e) \subseteq \omega$ and infinitely many stages such that $\widehat{E}(e)$ is precisely the set of numbers restrained by higher priority R strategies at such stages.

Assumption 2. The re-pairing activity of higher priority R strategies tends to infinity (over ω).

Basic Idea of the Proof: satisfying R_e.



$L(e,s+1)$ contains $n_0,\ldots,n_4,a$

$E(e,s+1)$, the restraint for R_e, contains $\{n : n < a\} \cup \{p(m) : m \in L(e,s+1)\}$

Conclusion of Proof.

Assuming the Assumptions 1-2 are true.

- R_e is satisfied and $E(e, s)$ stabilises over infinitely many stages s. Moreover for some such stage s, $E(e, s) \subseteq E(e, t)$for all stages $t \geq s$.
- Also the re-pairing activity of the R_e strategy tends to infinity.

Thus we can conclude that Assumptions 1-2 hold for $e + 1$ and, by induction, that $\{R_e\}_{e \in \omega}$ is satisfied and also that $\{Q_n\}_{n \in \omega}$ is satisfied, since any number n can only be re-paired by an R_i strategy if $i < n$.

Section Guide

Further Results.

Theorem

For any uniform Δ_2^0 class $\mathcal{F}$ there exists a computable linear order of order type $\omega + \zeta$ which is $\mathcal{F}$-rigid.

Remark

For any linear orders $\mathcal{L}$, $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ is of order type ω and $\mathcal{B}$ is of order type ζ, and $\mathcal{L} = \mathcal{A} + \mathcal{B}$, and automorphism f of $\mathcal{L}$, $f(z) = z$ for all $z \in A$ (the domain of $\mathcal{A}$). Moreover if f is a nontrivial automorphism, then $f(z) \neq z$ for all $z \in B$ (the domain of $\mathcal{B}$).

Further Results.

Note

The above theorem holds for a class of similar order types. For example for order type

$$\gamma_0\,\tau_0\,\gamma_1\,\tau_1\cdots\,\gamma_n\,\tau_n$$

with $n \geq 0$ and γ_i and τ_i being of order type ω and ζ respectively for all $i \leq n$.

THE END