Axiomatizing Resource Bounds for Measure

Xiaoyang Gu[1] Jack Lutz[2] Satyadev Nandakumar[3] Jim Royer[4]

[1]LinkedIn Corporation

[2]Iowa State University

[3]Indian Institute of Technology Kanpur

[4]Syracuse University

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2. Classical Measurability
3. Measure on the Cantor Space
4. Type-2 Computability
5. Resource-Bounded Measure on the Cantor Space

Resource bounded measure

- is a generalization of Lebesgue's measure theory, quantifying "sizes" of subsets of complexity classes.

Resource-Bounded Measure in Complexity Theory

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- Provides a generalization of the "probabilistic method" that works inside complexity classes.
- Gives fruitful hardness hypotheses (e.g. NP is a non-measure zero subset of exponential time) with plausible consequences.

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A resource bound Δ is a class of functions.

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  - Δ-measure:
  - Measure in a complexity class $R(\Delta)$.

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Main Contribution

Provide a simple and natural set of axioms with the following two properties.

- *Adequacy* – Any class Δ satisfying the axioms can be used as a resource bound for measure.

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Provide a simple and natural set of axioms with the following two properties.

- *Adequacy* – Any class Δ satisfying the axioms can be used as a resource bound for measure.
- *Generality* – The most extensively used resource bounds satisfy the axioms.

Classical Measurability

Definition

Let Ω be a set. An *outer measure* is a function $\phi : 2^{\Omega} \to [0, \infty]$ such that

- $\phi(\emptyset) = 0$.
- Monotonicity - $A \subseteq B \Leftarrow \phi(A) \leq \phi(B)$.
- Countable Subadditivity - $\phi\left(\cup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \phi(A_j)$.

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Note, this is defined for all subsets of Ω.

Carathéodory Definition of Measurable Set

Definition (Measurable Set)

Let $\phi : \Omega \to [0, \infty]$ be an outer measure. A set $E \subseteq \Omega$ is called *measurable* if for all sets $A \subseteq \Omega$,

$$\phi(A) = \phi(A \cap E) + \phi(A \cap E^c).$$

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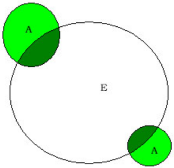

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Resource-Bounded Measure - Preliminaries

We focus on the binary alphabet $\Sigma = \{0, 1\}$. The set of finite strings is represented as $\{0, 1\}^*$. **C** is the Cantor Space, the space of all infinite binary sequences.

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A *probability measure* on **C** is a function $\nu : \{0, 1\}^* \to [0, 1]^*$ defined such that

- $\nu(\lambda) = 1$.
- For all finite strings w, $\nu(w) = \nu(w0) + \nu(w1)$.

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A *ν-martingale* is a function $d : \{0,1\}^* \rightarrow [0,\infty)$ such that

- $d(\lambda) \leq 1$.
- For all finite strings w,
  $d(w)\nu(w) = d(w0)\nu(w0) + d(w1)\nu(w1)$.

Probability and Martingales on **C**

Definition

Let ν be a probability measure, d be a ν-martingale, and $A \subseteq \{0,1\}^*$.

We say that *d covers A* if there is an n such that $d(A \upharpoonright n) \geq 1$. The *set covered by d* is $S^1[d] = \{A \mid d \text{ covers } A\}$.

We say that *d succeeds on A* if $\limsup_{n\to\infty} d(A \upharpoonright n) = \infty$. The *success set of d* is $S^\infty[d] = \{A \mid d \text{ succeeds on } A\}$.

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The first seems “feasible” but the second contains exponentially many terms in the *type-0 input*, $|x|$.
The issue: What is the “length” of a type-1 input?

- COMP: A function $f : \{0,1\}^m \to \{0,1\}$ is defined by *composition* from

$$g : \{0,1\}^m \to \{0,1\} \quad \text{and}$$
$$h : \{0,1\}^{m+1} \to \{0,1\}$$

if

$$f(\vec{x}) = h(g(\vec{x}), \vec{x}).$$

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- LRN: A function $f : \{0,1\}^m \to \{0,1\}$ is defined by *limited recursion on notation* from

$$\text{base} : \{0,1\}^{m-1} \to \{0,1\},$$
$$\text{ind}_0, \text{ind}_1 : \{0,1\}^{m+1} \to \{0,1\} \text{ and}$$
$$\text{bound} : \{0,1\}^m \to \{0,1\}$$

if

$$\begin{aligned} f(\vec{y}, \lambda) &= \text{base}(\vec{y}) \\ f(\vec{y}, xb) &= \text{ind}_b(f(\vec{y}, x), \vec{y}, x) \qquad \text{for } b \in \{0,1\} \\ |f(\vec{x})| &\leq |\text{bound}(\vec{x})| \end{aligned}$$

Recursive Characterization of P

Let $[f_0, f_1, \ldots, f_m; s_0, s_1, \ldots, s_n]$ denote the function algebra containing the initial functions $f_i, 0 \leq i \leq m$ and closed under the schema $s_j, 0 \leq j \leq n$.

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Theorem (Cobham, 1965)

$$\mathrm{P} = [0, s_0, s_1, \pi^i_n, \#; COMP, LRN],$$

where

$$s_0(x) = x0, \qquad s_1(x) = x1$$

are the string successor functions, π^i_n *the projection functions, and*

$$\#(x, y) = 2^{|x||y|}$$

(obeys $|\#(x, y)| = |x||y|$*).*

Type-2 Computability - Basic Feasible Functionals

Kapron and Cook [KC'91] defined Type-2 analogues of Cobham's schema:

- $\textsc{Expn}_2$:

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- $\textsc{Expn}_2$: F is defined from G, H, K by *expansion* if for all $\vec{f}, \vec{g}, \vec{x}, \vec{y}$,
$$F(\vec{f}, \vec{g}, \vec{x}, \vec{y}) = G(\vec{f}, \vec{x}).$$

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- $\textsc{Expn}_2$: Type-2 analogue of Projection
- $\textsc{Comp}_2$: F is defined from $G_0, \ldots, G_m, H$ by *functional composition* if for all $\vec{f}, \vec{x}$,

$$F(\vec{f}, \vec{x}) = H(G_0(\vec{f}, \vec{x}), \ldots, G_m(\vec{f}, \vec{x}), \vec{f}, \vec{x}).$$

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- $\textsc{Lrn}_2$: Type-2 Limited Recursion on Notation

Type-2 Computability - Basic Feasible Functionals

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- $\textsc{Expn}_2$: Type-2 analogue of Projection
- $\textsc{Comp}_2$: Type-2 analogue of Composition
- $\textsc{Lrn}_2$: Type-2 Limited Recursion on Notation F is defined from $\textsc{Base}, \textsc{Ind}_0, \textsc{Ind}_1, \textsc{Bound}$ by *limited recursion on notation* if for all $\vec{f}, \vec{x}$ and type-0 inputs w,

$$F(\vec{f}, \vec{x}, \lambda) = \textsc{Base}(\vec{f}, \vec{x})$$

$$F(\vec{f}, \vec{x}, wb) = \textsc{Ind}_b(F(\vec{f}, \vec{x}, w), \vec{f}, \vec{x}, w) \qquad \text{for } b \in \{0, 1\}$$

$$|F(\vec{f}, \vec{x}, w)| \leq |\textsc{Bound}(\vec{f}, \vec{x}, w)|$$

Type-2 Feasibility - Basic Feasible Functionals

Definition (Kapron, Cook '91)

Let X be a class of type-2 functionals.

$$\mathrm{BFF}(X) = [X, Ap, \mathrm{P}; \textsc{Comp}_2, \textsc{Expn}_2, \textsc{Lrn}_2],$$

where

$$Ap(f, x) = f(x)$$

is the application functional. $\mathrm{BFF} \triangleq \mathrm{BFF}(\emptyset)$.

Robustness - BFF

Definition (Kapron and Cook '91)

For any $f : \{0,1\}^* \to \{0,1\}^*$, the *length* of f is the function $|f| : \mathbb{N} \to \mathbb{N}$ defined by

$$|f|(n) = \max_{y \text{ prefix of } x} |f(y)|.$$

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Definition

Let $i \in \mathbb{Z}^+$.

- First-Order variables are elements of the set $\{n_1, n_2, \ldots\}$.
- Second-Order variables are elements of the set $\{L_1, L_2, \ldots\}$.

Second-order Polynomials are defined inductively.

- First-Order variables are second-order polynomials.
- If P, Q are second-order polynomials, then so is $P + Q, P \cdot Q$, and $L(P)$.

BFF - robustness

Theorem (Kapron, Cook '91)

A type-2 functional F is basic feasible if and only if there is an OTM M and a second-order polynomial P such that M computes F, and for all $\vec{f}$ and $\vec{x}$, the running time of $M(\vec{f}, \vec{x})$ is at most $P(|\vec{f}|, |\vec{x}|)$.

Type-2 Computability - Generaliziations to Quasi-Feasible

Let

$$\begin{aligned} g_0(n) &= 2n \\ g_i(n) &= 2^{g_{i-1}(\log n)} \end{aligned}$$

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For example,

$$g_1(n) = n^2$$
$$g_3(n) = n^{-1+\log n^{-1+\log\log n}}$$

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Definition (Gu, Lutz, Nandakumar, Royer '11)

Let $i \in \mathbb{Z}^+$.

- First-Order variables are elements of the set $\{n_1, n_2, \dots\}$.
- Second-Order variables are elements of the set $\{L_1, L_2, \dots\}$.

Second-order quasi^i-Polynomials are defined inductively.

- First-Order variables are second-order polynomials.
- If P, Q are second-order polynomials, then so is $P + Q, P \cdot Q, L(P)$ and $g_i(P)$.

Type-2 Computability - Quasi-Feasible Functionals

Definition (Gu, Lutz, Nandakumar, Royer '11)

Let X be a class of type-2 functionals.

$$\mathrm{BFF}_i(X) = [X, Ap, \mathrm{P}, 1^{g_i(x)}; \textsc{Comp}_2, \textsc{Expn}_2, \textsc{Lrn}_2],$$

where

$$Ap(f, x) = f(x)$$

is the application functional. $\mathrm{BFF}_i \triangleq \mathrm{BFF}_i(\emptyset)$.

Robustness - Quasi-Feasible Functionals

Theorem (Gu, Lutz, Nandakumar, Royer '11)

Let $i \in \mathbb{Z}^+$. A functional F is BFF_i if and only if there is an OTM M and a second-order quasi^i-polynomial P such that M computes F and for all $\vec{f}, \vec{x}$, the running time of $M(\vec{f}, \vec{x})$ is at most $P(|\vec{f}|, |\vec{x}|)$.

Space-Feasible Functionals

D. B. Thompson [T72] has a characterization of PSPACE as a function algebra, using (type-1) bounded recursion (on value).

- $\textsc{Br}_2$: A functional F is defined from $\textsc{Base}, \textsc{Ind}, \textsc{Bound}$ by *bounded recursion* (on value) if for all $\vec{f}, \vec{x}, n$,

$$
\begin{aligned}
F(vecf, \vec{x}, 0) &= \textsc{Base}(\vec{f}, \vec{x}) \\
F(\vec{f}, \vec{x}, n+1) &= \textsc{Ind}(F(\vec{f}, \vec{x}, n), \vec{f}, \vec{x}, n) \\
F(\vec{f}, \vec{x}, n) &\leq \textsc{Bound}(\vec{f}, \vec{x}, n).
\end{aligned}
$$

Robustness of Space-Feasibile Functionals

Definition (Gu, Lutz, Nandakumar, Royer '11)

Let X be a class of type-2 functionals.

$$\mathrm{BFSF}_i(X) = [X, Ap, \mathrm{P}, 1^{g_i(x)}; \textsc{Comp}_2, \textsc{Expn}_2, \textsc{Br}_2],$$

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is the application functional. $\mathrm{BFSF}_i \triangleq \mathrm{BFSF}_i(\emptyset)$.

Theorem (Gu, Lutz, Nandakumar, Royer '11)

A functional F is quasi^i-space feasible if and only if there is an OTM M and a second-order quasi^i polynomial P such that M computes F and for all $\vec{f}, \vec{x}$, the running space consumed by M is at most $P(|\vec{f}|, |\vec{x}|)$.

Resource Bounds

Definition (Gu, Lutz, Nandakumar, Royer '11)

A resource bound is a class Δ of type no more than 2 that is closed under the schema of $\textsc{Bff}$.

Theorem (Gu, Lutz, Nandakumar, Royer '11)

The following hold.

1. *For all numbers i, BFF_i is a resource bound.*
2. *Let $i \geq 1$, $k \geq 2$ be numbers, K^k be the canonical Σ_k^P complete language, and χ_k be its characterestic function. Then, $BFF_i(\{\chi_{k-1}\})$ is a resource bound.*
3. *For all numbers i, $\textsc{Bfsf}_i$ is a resource bound.*

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Computable Probability and Martingales

Definition

For any discrete domain D, a *computation* of a function $f : D \to \mathbb{R}$ is a function $\hat{f} : \mathbb{N} \times D \to \mathbb{Q}$ such that

$$|\hat{f}(n, x) - f(x)| \leq 2^{-n}.$$

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Definition

Let Δ be a resource-bound. A *Δ-probability measure* on **C** is a probability measure ν on **C** such that

- ν is Δ-computable
- There is a Δ-computable function $l : \mathbb{N} \to \mathbb{N}$ such that for all strings w, either $\nu(w) = 0$ or $\nu(w) \geq 2^{-l(w)}$.

Constructors and $R(\Delta)$

Definition

A *constructor* is a function $\delta : \{0,1\}^* \rightarrow \{0,1\}^*$ such that x is a proper prefix of $\delta(x)$ for all strings x. The *result of δ* is the unique language $R(\delta)$ such that $\delta^k(\lambda)$ is a prefix of $R(\delta)$.

If Δ is a resource bound, then the *result class* $R(\Delta)$ is the set of all languages $R(\delta)$ such that $\delta \in \Delta$.

e.g. If $\Delta = \mathrm{P}$, then $R(\Delta) = \mathrm{EXP}$.

Martingale Splitting Operators and RBM

Definition (Lutz)

Let X^+ and X^- be disjoint subsets of $\mathbf{C}$. A ν-martingale splitting operator for (X^+, X^-) is a type-2 functional

$$\Phi : \mathbb{N} \times D_\nu \to D_\nu \times D_\nu$$

such that

$$\Phi(r, d) = (\Phi_r^+(d), \Phi^-(d))$$

has the following properties for all numbers r and $d \in D_\nu$.

1. $X^+ \cap S^1[d] \subseteq S^1[\Phi_r^+(d)]$
2. $X^- \cap S^1[d] \subseteq S^1[\Phi_r^-(d)]$
3. $\Phi_r^+(d)(\lambda) + \Phi_r^-(d)(\lambda) \leq d(\lambda) + 2^{-r}$

Splitting Operators and RBM

Definition (Lutz)

Let $X \subseteq \mathbf{C}$. A $\Delta - \nu$ measurement of X is a $\Delta - \nu$ splitting operator for (X, X^c). We say X is $\Delta - \nu$ *measurable* if X has such a measurement.

Definition

If $X \subseteq \mathbf{C}$ is $\Delta - \nu$ measurable, then the Δ- ν measure of X is

$$\inf_{r \in \mathbb{N}} \Phi_r^+(\mathbf{1})(\lambda).$$

Research Directions

1. There are measure notions not captured by this axiomatization - e.g. the measure 0 on P developed by Moser, the notion of measure 0 on doubly exponential developed by Harkins and Hitchcock, and measure 0 on probabilistic classes developed by Moser.
   Is there an axiomatization whose measure 0 fragment coincides with these theories?
2. Lambov proved that BFF functionals over $\mathbb{R}^n$ compute exactly the class of Ko's polytime computable functions over $\mathbb{R}^n$.
   Does this extend to the new classes of type-2 functionals?