Expected utility operators and their possibilistic indicators

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Introduction

Uncertainty situations are traditionally modeled by probability theory.

In risk theory there are 2 important problems:

(I) when a risk situation is riskier than others

(II) if 2 agents face a risk situation when we can say that an agent is more risk averse than another

Risk situations are represented by random variables, and the attitude of an agent towards risk is represented by a utility function.

Possibility theory (Zadeh 1978) is an alternative to probability theory in the treatment of situations of uncertainty (in particular, risk situations).

The transition from probabilistic models to possibilistic models is done by:

- random variables are replaced with possibility distributions
- probabilistic indicators are replaced with possibilistic indicators
    - probabilistic expected value $\rightarrow$ possibilistic expected value
    - probabilistic variance $\rightarrow$ possibilistic variance
    - probabilistic covariance $\rightarrow$ possibilistic covariance

Fuzzy numbers are the most studied class of possibility distributions.

For fuzzy numbers there is one notion of expected value, but several notions of variance and covariance.

The objectives of this paper are:

• the introduction of expected utility operators as an abstract framework in the treatment of possibilistic variances and covariances

• to develop a theory of risk aversion in this abstract framework (problem II)

Fuzzy Numbers

A *fuzzy subset* of $\mathbf{R}$ is a function $A : \mathbf{R} \to [\mathbf{0}, \mathbf{1}]$. The support of a fuzzy subset A of $\mathbf{R}$ is $supp(A) = \{x \in \mathbf{R} | \mathbf{A}(\mathbf{x}) > \mathbf{0}\}$.

Let $\gamma \in [0,1]$. The $\gamma-$*level set* $[A]^\gamma$ of A is defined by

$$[A]^\gamma = \begin{cases} \{x \in \mathbf{R} | \mathbf{A}(\mathbf{x}) \geq \gamma\} & \text{if} \quad \gamma > 0 \\ cl(supp(A)) & \text{if} \quad \gamma = 0 \end{cases}$$

($cl(supp(A))$ is the topological closure of $supp(A)$.)

A fuzzy subset A of $\mathbf{R}$ is called *fuzzy number* if it verifies the following conditions:

- $A : \mathbf{R} \to [\mathbf{0}, \mathbf{1}]$ is a continuous function;
- A is normal, i.e. $A(x) = 1$ for some $x \in \mathbf{R}$;
- A is fuzzy convex, i. e. $[A]^\gamma$ is convex for any $\gamma \in [0,1]$;
- $supp(A)$ is a bounded subset of $\mathbf{R}$.

If A is a fuzzy number, then for any $\gamma \in [0,1]$, $[A]^\gamma$ is a compact and convex subset of $\mathbf{R}$. Then any γ–level set of A has the form $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0,1]$.

Let A, B be two fuzzy numbers and $\lambda \in \mathbf{R}$. Then the fuzzy numbers $A + B$ and λA are defined using Zadeh's extension principle :

$(A + B)(z) = \sup\{A(x) \wedge B(y) | x + y = z\}$ for any $z \in \mathbf{R}$

$(\lambda A)(y) = \sup\{A(x) | \lambda x = y\}$ for any $y \in \mathbf{R}$.

Assume that $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ and $[B]^\gamma = [b_1(\gamma), b_2(\gamma)]$ for any $\gamma \in [0, 1]$. Then

$[A + B]^\gamma = [a_1(\gamma) + b_1(\gamma), a_2(\gamma) + b_2(\gamma)]$

$[\lambda A]^\gamma = [\lambda a_1(\gamma), \lambda a_2(\gamma)]$ if $\lambda \geq 0$ and $[\lambda A]^\gamma = [\lambda a_2(\gamma), \lambda a_1(\gamma)]$ if $\lambda < 0$.

A triangular fuzzy number $A = (a, \alpha, \beta)$, with $a \in \mathbf{R}$ and $\alpha, \beta \geq 0$ is defined by

$$A(t) = \begin{cases} 1 - \frac{a-t}{\alpha} & \text{if} \quad a - \alpha \leq t \leq a \\ 1 - \frac{t-a}{\beta} & \text{if} \quad a \leq t \leq a + \beta \\ 0 & \text{otherwise} \end{cases}$$

Indicators of fuzzy numbers

A function $f : [0,1] \to \mathbf{R}$ is a *weighting function* if it is non–negative, monotone increasing and verifies the normality condition $\int_0^1 f(\gamma)d\gamma = 1$.

Let A be a fuzzy number such that $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for any $\gamma \in [0,1]$. According to [9] the f–weighted *possibilistic expected value* of A is defined by

$$E(f,A) = \frac{1}{2}\int_0^1 [a_1(\gamma) + a_2(\gamma)] f(\gamma) d\gamma. \qquad (1)$$

For $f(\gamma) = 2\gamma$, $\gamma \in [0,1]$, $E(f,A)$ is the possibilistic mean value of [2].

We introduce now three notions of f–weighted possibilistic variances:

$$Var_1(f,A) = \frac{1}{12}\int_0^1 [a_2(\gamma) - a_1(\gamma)]^2 f(\gamma) d\gamma \qquad (2)$$

$$Var_2(f,A) = \frac{1}{2}\int_0^1 ([a_1(\gamma) - E(f,A)]^2 + [a_2(\gamma) - E(f,A)]^2) f(\gamma) d\gamma \qquad (3)$$

$$Var_3(f,A) = \int_0^1 [\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} (x - E(f,A))^2 dx] f(\gamma) d\gamma \tag{4}$$

Remark 1 *$Var_1(f,A)$ and $Var_2(f,A)$ are generalizations of the two variances $Var(A)$ and $Var\prime(A)$ introduced by Carlsson and Fullér in [2] for $f(\gamma) = 2\gamma$, $\gamma \in [0,1]$. $Var_1(f,A)$ was originally introduced by Fullér and Majlender in [9] as $\frac{1}{4}\int_0^1 [a_2(\gamma) - a_1(\gamma)] f(\gamma) d\gamma$ and in form (2) by Carlsson, Fullér and Majlender in [5]. $Var_2(f,A)$ appears in [18] and $Var_3(f,A)$ in [10].*

In [10] the following calculation formulas for $Var_3(f,A)$ were proved.

$$Var_3(f,A) = \frac{1}{3}\int_0^1 [a_1^2(\gamma) + a_2^2(\gamma) + a_1(\gamma)a_2(\gamma)] f(\gamma) d\gamma - E^2(f,A) \tag{5}$$

$$Var_3(f,A) = 4Var_1(f,A) - E^2(f,A) + \int_0^1 a_1(\gamma)a_2(\gamma) f(\gamma) d\gamma \tag{6}$$

The following calculation formula for $Var_2(f, A)$ can be established:

$$Var_2(f, A) = \frac{1}{2} \int_0^1 [a_1^2(\gamma) + a_2^2(\gamma)] f(\gamma) d\gamma - E^2(f, A) \quad (7)$$

Applying (5)–(7) the following relation is obtained

$$Var_3(f, A) = Var_2(f, A) - 2Var_1(f, A) \quad (8)$$

Expected utility operators

We define the notion of expected utility operator and variance associated with it.

Let $\mathcal{F}$ be the set of fuzzy numbers and $\mathcal{C}(\mathbf{R})$ the set of continuous functions $g : \mathbf{R} \to \mathbf{R}$. We consider a subset $\mathcal{U}$ of $\mathcal{C}(\mathbf{R})$ with the properties:

(U_1) $\mathcal{U}$ contains constant functions and first and second order polynomial functions.

(U_2) If $\alpha, \beta \in \mathbf{R}$ and $g, h \in \mathcal{U}$ then $\alpha g + \beta h \in \mathcal{U}$.

We fix a weighting function f and a family of functions $\mathcal{U}$ with properties (U_1) and (U_2).

If $\lambda \in \mathbf{R}$ then we denote with $\bar{\lambda} : \mathbf{R} \to \mathbf{R}$ the constant function taking value λ.

Definition 2 *An (f–weighted)* expected utility operator *is a function $T : \mathcal{F} \times \mathcal{U} \to \mathbf{R}$ such that for any $\lambda, \mu \in \mathbf{R}$, $g, h \in \mathcal{U}$ and $A \in \mathcal{F}$ the following properties take place:*

(a) $T(A, 1_{\mathbf{R}}) = E(f, A)$; (b) $T(A, \bar{\lambda}) = \lambda$; (c) $T(A, \lambda g + \mu h) = \lambda T(A, g) + \mu T(A, h)$; (d) If $g \leq h$ then $T(A, g) \leq T(A, h)$.

The real number $T(A, g)$ is called *generalized possibilistic expected utility* of A with respect to f and g.

$T(A, \lambda g + \mu) = \lambda T(A, g) + \mu$ follows from axioms (b) and (c). In particular we have $T(A, -g) = -T(A, g)$.

Definition 3 *We consider the function* $g(x) = (x - E(f, A))^2$ *for any* $x \in \mathbf{R}$. *Then the* $T-$*variance* $Var_T(A)$ *of* A *is defined by* $Var_T(A) = T(A, g)$.

Since $g \geq 0$, $Var_T(A) \geq 0$ follows from axiom (d).

Example 4 *Let* $\mathcal{U} = \mathcal{C}(\mathbf{R})$. *For any* $g \in \mathcal{C}(\mathbf{R})$ *we denote*

$$E_2(f, g(A)) = \frac{1}{2} \int_0^1 [g(a_1(\gamma)) + g(a_2(\gamma))] f(\gamma) d\gamma \qquad (9)$$

The function $T_2 : \mathcal{F} \times \mathcal{C}(\mathbf{R}) \rightarrow \mathbf{R}$ *defined by* $T_2(A, g) = E_2(f, g(A))$ *for any* $A \in \mathcal{F}$ *and* $g \in \mathcal{C}(\mathbf{R})$ *is an expected utility operator. The variance associated with* T_2 *is*

$$Var_{T_2}(A) = Var_2(f, A) \qquad (10)$$

Example 5 *For any $g \in \mathcal{C}(\mathbf{R})$ we denote*

$$E_3(f, g(A)) = \int_0^1 [\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} g(x)dx] f(\gamma) d\gamma \tag{11}$$

In some cases the right hand side member of (11) can take infinite value. Let $\mathcal{U}_3$ be the set of functions $g \in \mathcal{C}(\mathbf{R})$ for which the integral of (11) is finite. $\mathcal{U}_3$ verifies (U_1) and (U_2). The function $T_3 : \mathcal{F} \times \mathcal{U}_3 \to \mathbf{R}$ defined by $T_3(A, g) = E_3(f, g(A))$ for any $A \in \mathcal{F}$ and $g \in \mathcal{U}_3$ is an expected utility operator and $Var_{T_3}(A) = Var_3(f, A)$.

Proposition 6 *If T, S are expected utility operators and α, β are two real numbers with $\alpha + \beta = 1$ then $U = \alpha T + \beta S$ is an expected utility operator and $Var_U(A) = \alpha Var_T(A) + \beta Var_S(A)$.*

Open problem Is there any expected utility operator T_1 such that $Var_{T_1}(A) = Var_1(f, A)$ for any $A \in \mathcal{F}$?

Possibilistic covariances

In this section we will associate a possibilistic covariance with each expected utility operator. Various possibilistic covariances from [5], [9], [10], [19] will be found as particular cases.

Let f be a weighting function and $T : \mathcal{F} \times \mathcal{U} \to \mathbf{R}$ be an expected utility operator. We consider two fuzzy numbers A, B such that $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ and $[B]^\gamma = [b_1(\gamma), b_2(\gamma)]$ for any $\gamma \in [0, 1]$.

In [5] the following possibilistic covariances were introduced:

$$Cov_1(f, A, B) = \frac{1}{12}\int_0^1 [a_2(\gamma) - a_1(\gamma)][b_2(\gamma) - b_1(\gamma)]f(\gamma)d\gamma$$

$$Cov_2(f, A, B) = \frac{1}{2}\int_0^1 [(a_1(\gamma) - E(f, A))(b_1(\gamma) - E(f, B)) + (a_2(\gamma) - E(f, A))(b_2(\gamma) - E(f, B))]f(\gamma)d\gamma$$

Obviously $Var_1(f, A) = Cov_1(f, A, A)$ and $Var_2(f, A) = Cov_2(f, A, A)$.

Proposition 7 *(i)* $Cov_2(f, A, B) = \frac{1}{2}\int_0^1 [a_1(\gamma)b_1(\gamma) + a_2(\gamma)b_2(\gamma)]$

$f(\gamma)d\gamma - E(f, A)E(f, B)$

(ii) $Cov_2(f, A, B) = 6Cov_1(f, A, B) - E(f, A)E(f, B) + \frac{1}{2}\int_0^1 [a_1(\gamma)b_2(\gamma) + a_2(\gamma)b_1(\gamma)]f(\gamma)d\gamma$

Equality (8) from Section 3 suggests a third possibilistic covariance

$Cov_3(f, A, B) = Cov_2(f, A, B) - 2Cov_1(f, A, B)$

One notices that $Var_3(f, A) = Cov_3(f, A, A)$.

Proposition 8 $Cov_3(f, A, B) = 4Cov_1(f, A, B) - E(f, A)E(f, B) + \frac{1}{2}\int_0^1 [a_1(\gamma)b_2(\gamma) + a_2(\gamma)b_1(\gamma)]f(\gamma)d\gamma$

Proposition 9 $Var_i(f, A+B) = Var_i(f, A) + Var_i(f, B) + 2Cov_i(f, A, B)$, $i = 1, 2, 3$

Definition 10 *The T–covariance $Cov_T(A, B)$ of fuzzy numbers A and B is defined by*

$$Cov_T(A, B) = \frac{1}{2}[Var_T(A+B) - Var_T(A) - Var_T(B)] \quad (12)$$

Since $A + B = B + A$ from (12) we get $Cov_T(A, B) = Cov_T(B, A)$.

In the following we will compute the T–covariances corresponding to expected utility operators T_2 and T_3 defined in the previous section.

Proposition 11 *Let T_2 be the expected utility operator from Example 4. Then*

$$Cov_{T_2}(A, B) = Cov_2(f, A, B) \quad (13)$$

Proposition 12 $Cov_{T_2}(A, B) = \frac{1}{2}\int_0^1 [a_1(\gamma)b_1(\gamma) + a_2(\gamma)b_2(\gamma)]$

$f(\gamma)d\gamma - E(f, A)E(f, B)$

Proposition 13 *Let T_3 be the expected utility operator from Example 5. Then*

$Cov_{T_3}(A,B) = 4Cov_1(f,A,B) - E(f,A)E(f,B) + \frac{1}{2}\int_0^1 [a_1(\gamma)b_2(\gamma)$

$+a_2(\gamma)b_1(\gamma)]f(\gamma)d\gamma$

Proposition 14 $Cov_{T_3}(A,B) = Cov_3(f,A,B)$

Proposition 15 *Let T, S be two expected utility operators and $\alpha, \beta \in \mathbf{R}$ such that $\alpha + \beta = 1$. If $U = \alpha T + \beta S$ then $Cov_U(A,B) = \alpha Cov_T(A,B) + \beta Cov_S(A,B)$.*

Let f be a weighting function and $T : \mathcal{F} \times \mathcal{U} \to \mathbf{R}$ be an expected utility operator. If $g : \mathbf{R} \to \mathbf{R}$ is the function $g(x) = x^2$ for any $x \in \mathbf{R}$ then we denote $T(A,g) = T(A,x^2)$.

Lemma 16 *For any fuzzy number A, $Var_T(A) = T(A,x^2) - E^2(f,A)$.*

In general the equality $Var_T(A) = Cov_T(A,A)$ is not true. The following result characterizes those expected utility operators for which this equality is true.

Proposition 17 *The following assertions are equivalent:*

(i) For any $A \in \mathcal{F}$, $T(2A,x^2) = 4T(A,x^2)$

(ii) For any $A \in \mathcal{F}$, $Var_T(2A) = 4Var_T(A)$

(iii) For any $A \in \mathcal{F}$, $Var_T(A) = Cov_T(A,A)$

Remark 18 *Expected utility operators T_2, T_3 verify the equivalent properties of the previous proposition.*

Proposition 19 *The following assertions are equivalent:*

(i) For any $A, B, C \in \mathcal{F}$, $Cov_T(A+B, C) = Cov_T(A, C) + Cov_T(B, C)$;

(ii) For any $A, B, C \in \mathcal{F}$, $Var_T(A+B+C) = Var_T(A+B) + Var_T(B+C) + Var_T(C+A) - Var_T(A) - Var_T(B) - Var_T(C)$;

(iii) For any $A, B, C \in \mathcal{F}$, $T(A+B+C, x^2) = T(A+B, x^2) + T(B+C, x^2) + T(C+A, x^2) - T(A, x^2) - T(B, x^2) - T(C, x^2)$.

Open problem Is there any expected utility operator T_1 such that $Cov_{T_1}(A, B) = Cov_1(f, A, B)$ for any $A, B \in \mathcal{F}$?

Possibilistic risk aversion

Probability theory of risk aversion (Arrow [1], Pratt [15]) has as central notion the risk premium. the risk premium is "the maximum amount by which the agent is willing to decrease the expected return from the lottery ticket to have a sure return" ([12], p. 19).

In papers [10], [11] 2 notions of possibilistic risk premium were introduced, based on 2 notions of possibilistic expected utility.

In this section we will define a notion of possibilistic risk premium in an abstract framework determined by the following elements:

• a weighting function $f : [0, 1] \to \mathbf{R}$

• an expected utility operator $T : \mathcal{F} \times \mathcal{U} \to \mathbf{R}$

• a fuzzy number A

• a utility function $u : \mathbf{R} \to \mathbf{R}$ twice differentiable, strictly concave and strictly increasing

Definition 20 *The possibilistic risk premium* $\rho = \rho(f, T, A, u)$ *associated with the quadruple* (f, T, A, u) *is defined by*

(1) $u(E(f, A) - \rho) = T(A, u)$

Since the utility function u is injective, the possibilistic risk premium ρ is uniquely determined by (1).

Remark 21 *The presence of the expected utility operator T in Definition 20 confers an increased degree of generality to the notion of possibilistic risk premium $\rho(f, T, A, u)$. By the particularization of T various types of possibilistic risk premiums result.*

If $T = T_2$ we obtain the possibilistic risk premium from [10] and if $T = T_3$ we obtain the possibilistic risk premium from [11].

The possibilistic risk premium is an indicator of risk aversion of an agent represented by u in front of a situation of uncertainty characterized by the fuzzy number A.

Proposition 22 *An approximate solution of (1) is given by*

(2) $\rho \approx -\frac{1}{2} Var_T(A) \frac{u''(E(f,A))}{u'(E(f,A))}$

Example 23 *Assuming $T = T_2$, equation (1) becomes*

(3) $u(E(f, A) - \rho) = E_2(f, u(A))$.

By Proposition 22 and Example 4 the approximate solution ρ_2 of equation (3) has the form

(4) $\rho_2 \approx -\frac{1}{2} Var_2(f, A) \frac{u''(E(f,A))}{u'(E(f,A))}$

Example 24 *Assuming $T = T_3$, equation (1) becomes*

(5) $u(E(f, A) - \rho) = E_3(f, u(A))$.

By Proposition 22 and Example 5 the approximate solution ρ_3 of equation (5) has the form

(6) $\rho_3 \approx -\frac{1}{2} Var_3(f, A) \frac{u''(E(f,A))}{u'(E(f,A))}$.

Let T, S be two expected utility operators and $U = \alpha T + \beta S$ with $\alpha + \beta = 1$. We write equation (1) for T, S and u:

$$u(E(f, A) - \rho) = T(A, u)$$

$$u(E(f, A) - \rho) = S(A, u)$$

$$u(E(f, A) - \rho) = U(A, u)$$

Let ρ_T, ρ_S, ρ_U be the approximate solutions of the three equations given by Proposition 22.

Proposition 25 $\rho_U \approx \alpha \rho_T + \beta \rho_S$.

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