Polynomial-Time Set Recursion

Joint work with Arnold Beckmann and Sam Buss

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How can we generalise this notion to arbitrary sets? In other words:

When is a function $F : V \to V$ computable in "polynomial time"?

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Consider some standard models for polynomial-time computation:

Polynomial-Time Set Recursion

1. *Turing machines*

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3. Schemes

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3. Schemes

This works, using the work of Bellantoni-Cook!

Bellantoni-Cook Recursion

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Define functions

$$f(\vec{x}/\vec{y})$$

where $\vec{x}, \vec{y}$ are finite sequences of finite (binary) strings and the values of f are finite strings

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When performing primitive recursions, the "previous value" is placed on the Safe side:

$f(0, \vec{x}/\vec{y}) = g(\vec{x}/\vec{y})$

$f(z * i, \vec{x}/\vec{y}) = h_i(z, \vec{x}/\vec{y}, f(z, \vec{x}/\vec{y}))$, $i = 0$ or 1

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$$f(\vec{x}/\vec{y}) = h(k(\vec{x}/-)/l(\vec{x}/\vec{y}))$$

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On safe inputs string-length is increased by only a constant amount

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in the class; but we only have

$M(y, z/-)$ (multiplication of Normal Inputs)

and no function

$M(y/z) = y \times z,$

which has z as a Safe Input.

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We adapt the Bellantoni-Cook idea to set theory by combining the Gandy-Jensen *rudimentary set functions* with the set-theoretic analogue of Bellantoni-Cook safe recursion

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Basic Functions

$\Pi_i^{m,n}(x_1, \ldots, x_m / x_{m+1}, \ldots, x_{m+n}) = x_i \ (1 \leq i \leq m+n)$
$\mathrm{Pair}(-/a, b) = \{a, b\}$
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$f(\vec{x}/\vec{a}, y) = \cup\{g(\vec{x}/\vec{a}, z) \mid z \in y\}$

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Safe Composition

$f(\vec{x}/\vec{a}) = h(\vec{r}(\vec{x}/-)/\vec{s}(\vec{x}, \vec{a}))$

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For ordinals α, β, $\oplus(\alpha/\beta) = \beta + \alpha$

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Proposition

If $f(\vec{x}/\vec{y})$ is a safe-recursive set function then there is a polynomial function p_f on ordinals such that:

$$rank(f(\vec{x}/\vec{y})) \leq \max(rank(\vec{y})) + p_f(rank(\vec{x}))$$

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Theorem

For transitive T, $SR(T) = L^T_{rank(T)^\omega}$, where L^T is the L-hierarchy relativised to T.

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So $\mathrm{SR}(T)$ contains J^T_α for $\alpha < \mathrm{rank}(T)^\omega$ and therefore contains $J^T_{rank(T)^\omega} = L^T_{rank(T)^\omega}$

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So $L^T_{rank(T)^\omega}$ is closed under SR functions.

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So we conclude: $\mathrm{SR}(T) = L^T_{rank(T)^\omega}$ for transitive T

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Suppose that $f(\vec{x}/-)$ is SR. Then for some Σ_1 formula φ and some finite n we have:

$$f(\vec{x}, -) = y \text{ iff } SR_n(\vec{x}) \vDash \varphi(\vec{x}, y).$$

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Conversely, any function so defined is SR.

The SR Hierarchy

Analog of Jensen's J-hierarchy:

$\mathrm{SR}_1 = \mathrm{HF}$, the collection of hereditarily finite sets
$\mathrm{SR}_{\alpha+1} = \mathrm{SR}(\mathrm{SR}_\alpha)$ for $\alpha > 0$
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Corollary

For every α, $SR_{1+\alpha} = L_{\omega(\omega^\alpha)}$.

The SR Hierarchy

Analog of Jensen's J-hierarchy:

$\mathrm{SR}_1 = \mathrm{HF}$, the collection of hereditarily finite sets
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$L_\omega \subseteq L_{\omega^\omega} \subseteq L_{\omega(\omega^2)} \subseteq L_{\omega(\omega^3)} \subseteq \cdots$

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Thus the SR functions on ω-strings look like:

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These functions coincide with those computable by an infinite-time Turing machine in time ω^n for some finite n, and were considered by Deolalikar, Hamkins, Schindler, Welch and others.

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Interestingly, a similar class arises as the complexity of the first-order theory of the reals with $+$ and $<$ (STA$(*, \text{Exp}, \text{Linear})$)

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Fact: The safe recursive set functions are exactly those computable by some machine as above