On the Kolmogorov Complexity of Continuous Real Functions

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Outline

1. Kolmogorov Complexity: Finite Objects
2. Non-Finite Objects
3. Main Results
4. Summary and Future Work

Kolmogorov Complexity

- first introduced by Ray Solomonoff around 1960
- also known as *descriptive complexity*
- originally defined over *finite-objects*
- can be used to study how ‘compressible’ a *finite sequence of* 0*s and* 1*s* is

Kolmogorov Complexity: Example

- We do not always have to write down all the elements of a sequence to *describe* it.
- For instance, we can describe the following sequence of digits in fewer than 100 characters:

$$x = \underbrace{111\ldots1}_{100}$$

- In fact, one can simply describe x in fewer than 25 characters as *'a sequence of* 100 *ones'*.

What About Non-Finite Objects?

- Consider the compact interval $[0, 1]$ of real numbers.
- For each $x \in [0, 1]$, there is a Cauchy sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ such that:

$$\begin{cases} \forall n \in \mathbb{N} : |x_n - x| < 2^{-n} \\ \lim_{n \to \infty} \langle x_n \rangle_{n \in \mathbb{N}} = x \end{cases}$$

- The binary expansion of x gives us one choice for this sequence.
- $\pi^{-1} = \lim < 0.0_2, 0.01_2, 0.010_2, 0.0101_2, \ldots >$

Kolmogorov Complexity of Real Numbers: Cai and Hartmanis [CH94]

- As the binary expansion is one candidate, then:

$$\forall n \in \mathbb{N} : K(x_n) \leq n$$

- For the real number x, the Kolmogorov complexity $K_{\mathbb{R}}(x)$ can be defined as:

$$K_{\mathbb{R}}(x) := \frac{1}{2}\left(\liminf_{n\to\infty} \frac{K(x_n)}{n} + \limsup_{n\to\infty} \frac{K(x_n)}{n}\right)$$

- It should be clear that $\forall x \in [0,1] : 0 \leq K_{\mathbb{R}}(x) \leq 1$

Some properties of $K_{\mathbb{R}}$

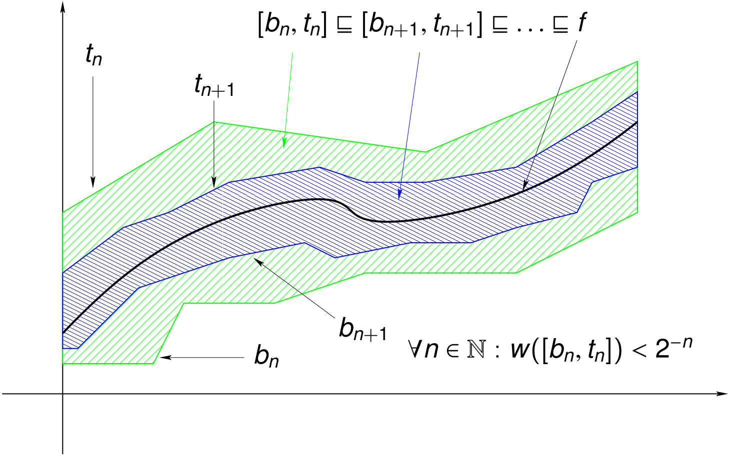
Cai and Hartmanis [CH94] prove that:

(i) For Lebesgue-almost every x in $[0,1]$, $K_{\mathbb{R}}(x) = 1$.

(ii) For every $t \in [0,1]$, the set $K_{\mathbb{R}}^{-1}(t)$ is uncountable and has Hausdorff dimension t.

(iii) The graph of $K_{\mathbb{R}}$ (which is a subset of $[0,1] \times [0,1]$) is a fractal.

Binary Representation of $f \in C[0,1]$ with its Function Enclosures as 'Digits'



Optimal Representation of Real Functions

Proposition

For every function $f \in C[0,1]$ there exists a representation $\hat{f}$ of f of minimal Kolmogorov complexity, i. e. for any other representation of f such as $\rho : \mathbb{N} \to \mathcal{B}$:

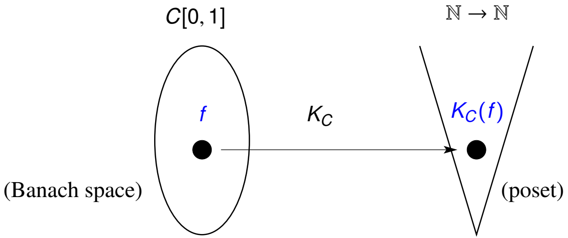
$$\forall n \in \mathbb{N} : K(\hat{f}(n)) \leq K(\rho(n))$$

Kolmogorov Complexity over Real Functions

Definition ($K_C(f)$)

Let $\hat{f} : \mathbb{N} \to \mathcal{B}$ be an optimal representation of $f \in C[0, 1]$. The *Kolmogorov complexity function* of f is defined as:

$$\begin{aligned} K_C(f) : \mathbb{N} &\to \mathbb{N} \\ n &\mapsto K(\hat{f}(n)) \end{aligned}$$



Existence of Functions with Arbitrarily High Growth Kolmogorov Complexity

Proposition

For any given $\theta : \mathbb{N} \to \mathbb{N}$, there exists a function f in $C[0,1]$ whose Kolmogorov complexity is above θ over infinitely many points. In other words:

$$\forall m \in \mathbb{N} : \exists n \geq m : K_C(f)(n) \geq \theta(n)$$

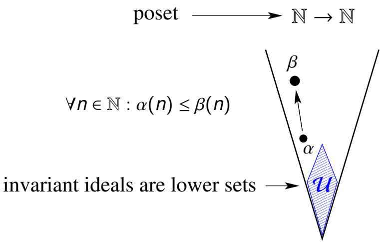
Existence of Functions with Arbitrarily High Growth Kolmogorov Complexity: Examples

For instance, in the previous proposition, by taking $\theta(n)$ to be:

1. 2^n, one can show that there exists a real function $f \in C[0,1]$ whose Kolmogorov complexity $K_C(f)$ is not dominated by any polynomial.
2. $n!$, one can show that there exists a real function $f \in C[0,1]$ whose Kolmogorov complexity $K_C(f)$ is not dominated by any exponential function.

The Asymptotic Behaviour of $K_C(f)$

- We are interested in the *asymptotic* behaviour of $K_C(f)$, specifically its *growth* rate.
- Our abstract notion of growth rate is presented via *invariant ideals* (see [Far11]).
- Informally, we call a set $\mathcal{U} \subseteq \mathbb{N}^{\mathbb{N}}$ an invariant ideal of $\mathbb{N}^{\mathbb{N}}$ if it has some good closure and translation-invariance properties.



Invariant Ideals

- Crucial: Invariant ideals have *countable bases*.
- Some examples of invariant ideals are the sets of functions from $\mathbb{N}$ to $\mathbb{N}$ bounded by:
  - polynomials with natural number coefficients
  - exponential functions with natural number parameters
- Invariant ideals form a hierarchy with conceptual similarities to the *Grzegorczyk hierarchy*.

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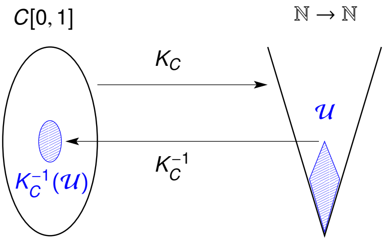
Some Notations

- We reserve the symbol $\mathcal{U}$ to denote an invariant ideal.
- Thus, $K_C^{-1}(\mathcal{U})$ will denote the set of functions $f \in C[0,1]$ whose Kolmogorov complexity function $K_C(f)$ is an element of $\mathcal{U}$.



Fundamental Proposition

Proposition

For any invariant ideal $\mathcal{U}$ the following are true:

(a) *the set $K_C^{-1}(\mathcal{U})$ is closed under arithmetic operations.*

(b) *the set $K_C^{-1}(\mathcal{U})$ is an F_σ set, i. e. it is the union of a countable family of closed sets.*

(c) *the set $K_C^{-1}(\mathcal{U})$ is Borel.*

(d) *let $\mathcal{U}^c$ denote the complement of $\mathcal{U}$, then:*

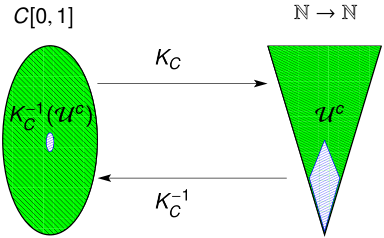
$$\forall f \in K_C^{-1}(\mathcal{U}^c),\, g \in K_C^{-1}(\mathcal{U}),\, \tau \in \mathbb{R} \setminus \{0\} : (\tau f + g) \in K_C^{-1}(\mathcal{U}^c)$$

Main Theorem

Theorem

For any invariant ideal $\mathcal{U}$, the set $K_C^{-1}(\mathcal{U}^c)$ is a prevalent subset of $C[0,1]$.

Informally, 'almost all' [HSY92] continuous real functions have 'very high growth' Kolmogorov complexity.



An Asymptotic Bound on the Kolmogorov Complexities of TTE-Computable Functions

- We can construct a function $\theta : \mathbb{N} \to \mathbb{N}$ in such a way that it can be regarded as an asymptotic bound on the Kolmogorov complexity of every *TTE-computable* $f \in C[0, 1]$.
- In other words, if $\mathcal{U}_\theta$ denotes the smallest invariant ideal that includes θ:

Theorem

The Kolmogorov complexity function of every $f \in K_C^{-1}({\mathcal{U}_\theta}^c)$ is an asymptotic upper bound for the Kolmogorov complexity function of any computable function g in $\tilde{C}[0, 1]$, i. e. there exists an infinite set $J \subseteq \mathbb{N}$ such that $\forall j \in J : K_C(g)(j) \leq K_C(f)(j)$.

Summary

- We defined a notion of Kolmogorov complexity over $C[0,1]$.
- We proved that functions with arbitrarily high growth Kolmogorov complexity exist.
- In fact, we showed that 'almost all' functions in $C[0,1]$ have 'high growth' Kolmogorov complexity.
- An asymptotic bound on the Kolmogorov complexities of TTE-computable functions in $C[0,1]$ was given.
- We presented some topological and measure theoretic properties of some subsets of $C[0,1]$ that are defined in terms of Kolmogorov complexity classes in $\mathbb{N}^{\mathbb{N}}$.

Future Directions:

- Is there any meaningful constructive content in our results?
  - If yes, then the classical theory of $C[0,1]$ can offer useful contribution to the constructive theory of $\tilde{C}[0,1]$.
- We used a representation of $C[0,1]$ suitable for 'integration-like' second-order operators:
  - What about representations suitable for other operators?
  - Note that integration and *the maximum value operator* can both be defined in terms of the existential quantifier [Esc97, Sim98].

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