A remark on the existence of contact relations on Boolean algebras

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Happy 12^2th birthday, Canada!

Points vs aggregates

A.N. Whitehead: The Organization of Thought, 1917

Our space concepts are concepts of relations between things in space. Thus there is no such entity as a self-subsistent point. A point is merely the name for some peculiarity of the relations between the matter which is, in common language, said to be in space.

It follows from the relative theory that a point should be definable in terms of the relations between material things. So far as I am aware, this outcome of the theory has escaped the notice of mathematicians, who have invariably assumed the point as the ultimate starting ground of their reasoning. Many years ago I explained some types of ways in which we might achieve such a definition, and more recently have added some others. Similar explanations apply to time. Before the theories of space and time have been carried to a satisfactory conclusion on the relational basis, a long and careful scrutiny of the definitions of points of space and instants of time will have to be undertaken, and many ways of effecting these definitions will have to be tried and compared. This is an unwritten chapter of mathematics, in much the same state as was the theory of parallels in the eighteenth century.

In this connection I should like to draw attention to the analogy between time and space. In analysing our experience we distinguish events, and we also distinguish things whose changing

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H. de Vries: Compact Spaces and Compactifications

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COMPACT SPACES
AND COMPACTIFICATIONS

AN ALGEBRAIC APPROACH

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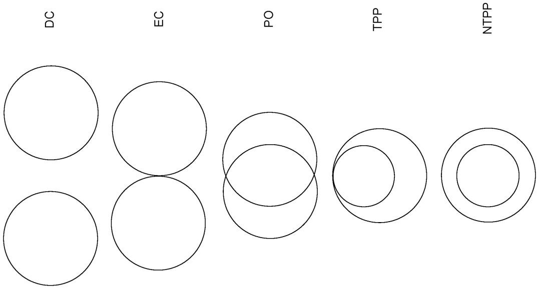
Algebras and representation

- de Laguna: Point, line and surface as sets of solids, 1922
- Nicod: Geometry in a sensible world, 1924
- Russell: Analysis of Matter, 1927

- Roeper, 1997 [9]
- Mormann, 1998, [6]
- Pratt, Schoop, 1998, 2002 [7, 8]
- Stell, 2000 [10]
- Düntsch et al, 2000, 2001, 2005 [3–5]
- Dimov, Vakarelov 2006 [1, 2]

The prototype of contact structures

- The complete Boolean algebra of regular closed sets ("regions") in the Euclidean plane.
- Regions a, b are in contact, written as aCb, if $a \cap b \neq \emptyset$.



Axioms for contact algebras

A Boolean contact algebra $\langle B, \mathscr{C} \rangle$ is a Boolean algebra B together with a binary relation $\mathscr{C}$ on B which satisfies for all $x, y, z \in B$

C_0. $0(-\mathscr{C})1$

C_1. $x \neq 0$ implies $x \mathscr{C} x$ (weak reflexivity)

C_2. $x \mathscr{C} y$ implies $y \mathscr{C} x$ (symmetry)

C_3. $x \mathscr{C} y$ and $y \leq z$ implies $x \mathscr{C} z$. (monotonicity)

C_4. $x \mathscr{C} (y + z)$ implies $(x \mathscr{C} y$ or $x \mathscr{C} z)$ (distributivity)

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- $x C_{\text{min}} y \iff x \cdot y \neq 0$ is the smallest contact relation.
- $x C_{\text{max}} y \iff x \neq 0$ and $y \neq 0$ is the largest contact relation.

Additional properties

C_{ext}. If $\{z : x\mathscr{C}z\} = \{z : y\mathscr{C}z\}$, then $x = y$. (extensionality)

C_{con}. If $x \neq 0$ and $x \neq 1$, then $x\mathscr{C}x^*$. (connectivity)

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- The smallest contact relation on B satisfies C_{ext}, but not C_{con}.
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Theorem

If $\langle B, \mathscr{C} \rangle$ satisfies C_0 – C_4 and C_{ext} and C_{con}, then B is atomless.

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If $\langle B, \mathscr{C} \rangle$ satisfies $C_0 - C_4$ and C_{ext} and C_{con}, then B is atomless.

Does every atomless Boolean algebra admit a contact relation that satisfies C_{ext} and C_{con}?

Theorem

For every atomless Boolean algebra D, there is a contact relation $\mathscr{C}$ on D such that $\langle D, \mathscr{C} \rangle$ satisfies C_{ext} and C_{con}.

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Auxiliary Lemma

Assume that C is a complete atomless Boolean algebra. Then, there is a pair (A, B) of disjoint dense subalgebras of C.

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Proof outline: Let D be atomless and C its completion. Choose A, B as in the lemma. Now consider these contact relations:

$\mathscr{C}_1 =$ overlap relation on A	(satisfies C_{ext})
$\mathscr{C}_2 =$ canonical extension of $\mathscr{C}_1$ to C	(satisfies C_{ext})
$\mathscr{C}_3 =$ restriction of $\mathscr{C}_2$ to B	(satisfies C_{ext} and C_{con})
$\mathscr{C}_4 =$ canonical extension of $\mathscr{C}_3$ to C	(satisfies C_{ext} and C_{con})
$\mathscr{C}_5 =$ restriction of $\mathscr{C}_4$ to D	(satisfies C_{ext} and C_{con})

Representation theorems

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2. *For every Boolean contact algebra which satisfies C_{ext} and C_{con} there is a connected compact weakly regular T_1 space X such that B is isomorphic to a subalgebra of* $\langle \mathrm{RegCl}(X), \mathscr{C}_w \rangle$ (Dimov and Vakarelov [1], Düntsch and Winter [5]).

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Mnogo blagodarya
Thank you
Dziękuję
Danke
Merci

[1] Dimov, G. and Vakarelov, D. (2006a). Contact algebras and region–based theory of space: A proximity approach – I. *Fundamenta Informaticae*, 74:209 – 249.

[2] Dimov, G. and Vakarelov, D. (2006b). Contact algebras and region–based theory of space: A proximity approach –II. *Fundamenta Informaticae*, 74:251 – 282.

[3] Düntsch, I., Wang, H., and McCloskey, S. (2000). Relation algebras in qualitative spatial reasoning. *Fundamenta Informaticae*, pages 229–248.

[4] Düntsch, I., Wang, H., and McCloskey, S. (2001). A relation algebraic approach to the Region Connection Calculus. *Theoretical Computer Science*, 255:63–83.

[5] Düntsch, I. and Winter, M. (2005). A Representation Theorem for Boolean Contact Algebras. *Theoretical Computer Science (B)*, 347:498–512.

[6] Mormann, T. (1998). Continuous lattices and Whiteheadian theory of space. *Logic and Logical Philosophy*, 6:35–54.

[7] Pratt, I. and Schoop, D. (1998). A complete axiom system for polygonal mereotopology of the real plane. *Journal of Philosophical Logic*, 27(6):621–658.

[8] Pratt-Hartmann, I. and Schoop, D. (2002). Elementary Polyhedral Mereotopology. *Journal of Philosophical Logic*, 31(5):469–498.

[9] Roeper, P. (1997). Region based topology. *Journal of Philosophical Logic*, 26:251–309.

[10] Stell, J. (2000). Boolean connection algebras: A new approach to the Region Connection Calculus. *Artificial Intelligence*, 122:111–136.

[11] Stone, M. (1936). The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, 40:37–111.