A remark on the existence of contact relations on Boolean algebras

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Happy 12\textsuperscript{2}th birthday, Canada!
Points vs aggregates

A.N. Whitehead: The Organization of Thought, 1917

Our space concepts are concepts of relations between things in space. Thus there is no such entity as a self-subsistent point. A point is merely the name for some peculiarity of the relations between the matter which is, in common language, said to be in space.

It follows from the relative theory that a point should be definable in terms of the relations between material things. So far as I am aware, this outcome of the theory has escaped the notice of mathematicians, who have invariably assumed the point as the ultimate starting ground of their reasoning. Many years ago I explained some types of ways in which we might achieve such a definition, and more recently have added some others. Similar explanations apply to time. Before the theories of space and time have been carried to a satisfactory conclusion on the relational basis, a long and careful scrutiny of the definitions of points of space and instants of time will have to be undertaken, and many ways of effecting these definitions will have to be tried and compared. This is an unwritten chapter of mathematics, in much the same sense as was the theory of parallels in the eighteenth century.

In this connection I should like to draw attention to the analogy between time and space. In analysing our experience we distinguish events, and we also distinguish things whose changing
Points vs aggregates

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H. de Vries: Compact Spaces and Compactifications

DR. H. DE VRIES

COMPACT SPACES AND COMPACTIFICATIONS

AN ALGEBRAIC APPROACH

ASSEN MCMXII

VAN GORcum & COMP, N.V. - DR. H. J. Prakke & H. M. G. Prakke
Algebras and representation

- de Laguna: Point, line and surface as sets of solids, 1922
- Nicod: Geometry in a sensible world, 1924
- Russell: Analysis of Matter, 1927
- Roeper, 1997 [9]
- Pratt, Schoop, 1998, 2002 [7, 8]
- Stell, 2000 [10]
- Dimov, Vakarelov 2006 [1, 2]
The prototype of contact structures

- The complete Boolean algebra of regular closed sets ("regions") in the Euclidean plane.
- Regions $a, b$ are in contact, written as $aCb$, if $a \cap b \neq \emptyset$. 
A Boolean contact algebra $\langle B, C \rangle$ is a Boolean algebra $B$ together with a binary relation $C$ on $B$ which satisfies for all $x, y, z \in B$

\begin{align*}
C_0. & \quad 0(-C)1 \\
C_1. & \quad x \neq 0 \text{ implies } xC x \quad \text{(weak reflexivity)} \\
C_2. & \quad xC y \text{ implies } yC x \quad \text{(symmetry)} \\
C_3. & \quad xC y \text{ and } y \leq z \text{ implies } xC z. \quad \text{(monotonicity)} \\
C_4. & \quad xC(y + z) \text{ implies } (xC y \text{ or } xC z) \quad \text{(distributivity)}
\end{align*}
A Boolean contact algebra $\langle B, C \rangle$ is a Boolean algebra $B$ together with a binary relation $C$ on $B$ which satisfies for all $x, y, z \in B$

$C_0$. $0(-C)1$

$C_1$. $x \neq 0$ implies $xCx$ (weak reflexivity)

$C_2$. $xCy$ implies $yCx$ (symmetry)

$C_3$. $xCy$ and $y \leq z$ implies $xCz$. (monotonicity)

$C_4$. $xC(y+z)$ implies $(xCy$ or $xCz)$ (distributivity)

- $xC_{\text{min}}y \iff x \cdot y \neq 0$ is the smallest contact relation.
- $xC_{\text{max}}y \iff x \neq 0$ and $y \neq 0$ is the largest contact relation.
**Additional properties**

\[ C_{\text{ext}} \]: If \( \{ z : x \mathcal{C} z \} = \{ z : y \mathcal{C} z \} \), then \( x = y \). \hspace{1cm} \text{(extensionality)}

\[ C_{\text{con}} \]: If \( x \neq 0 \) and \( x \neq 1 \), then \( x \mathcal{C} x^* \). \hspace{1cm} \text{(connectivity)}
Additional properties

\( C_{\text{ext}} \). If \( \{ z : x \notC z \} = \{ z : y \notC z \} \), then \( x = y \). \hspace{1cm} \text{(extensionality)}

\( C_{\text{con}} \). If \( x \neq 0 \) and \( x \neq 1 \), then \( x \notC x^* \). \hspace{1cm} \text{(connectivity)}

- The smallest contact relation on \( B \) satisfies \( C_{\text{ext}} \), but not \( C_{\text{con}} \).
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Additional properties

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**Theorem**

If \( \langle B, \mathcal{C} \rangle \) satisfies \( C_0 \) – \( C_4 \) and \( C_{\text{ext}} \) and \( C_{\text{con}} \), then \( B \) is atomless.
Additional properties

$C_{\text{ext}}$. If $\{z : x \leq z\} = \{z : y \leq z\}$, then $x = y$. \hfill (extensionality)

$C_{\text{con}}$. If $x \neq 0$ and $x \neq 1$, then $x \leq x^*$. \hfill (connectivity)

- The smallest contact relation on $B$ satisfies $C_{\text{ext}}$, but not $C_{\text{con}}$.
- The largest contact relation on $B$ satisfies $C_{\text{con}}$, but not $C_{\text{ext}}$.

Theorem

If $\langle B, \leq \rangle$ satisfies $C_0 - C_4$ and $C_{\text{ext}}$ and $C_{\text{con}}$, then $B$ is atomless.

Does every atomless Boolean algebra admit a contact relation that satisfies $C_{\text{ext}}$ and $C_{\text{con}}$?
Theorem

For every atomless Boolean algebra $D$, there is a contact relation $C$ on $D$ such that $\langle D, C \rangle$ satisfies $C_{\text{ext}}$ and $C_{\text{con}}$. 
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Auxiliary Lemma
Assume that $C$ is a complete atomless Boolean algebra. Then, there is a pair $(A, B)$ of disjoint dense subalgebras of $C$. 
Theorem

For every atomless Boolean algebra $D$, there is a contact relation $C$ on $D$ such that $\langle D, C \rangle$ satisfies $C_{\text{ext}}$ and $C_{\text{con}}$.

Auxiliary Lemma

Assume that $C$ is a complete atomless Boolean algebra. Then, there is a pair $(A, B)$ of disjoint dense subalgebras of $C$.

Proof outline: Let $D$ be atomless and $C$ its completion. Choose $A, B$ as in the lemma. Now consider these contact relations:

$C_1 = \text{overlap relation on } A$ (satisfies $C_{\text{ext}}$)

$C_2 = \text{canonical extension of } C_1 \text{ to } C$ (satisfies $C_{\text{ext}}$)

$C_3 = \text{restriction of } C_2 \text{ to } B$ (satisfies $C_{\text{ext}}$ and $C_{\text{con}}$)

$C_4 = \text{canonical extension of } C_3 \text{ to } C$ (satisfies $C_{\text{ext}}$ and $C_{\text{con}}$)

$C_5 = \text{restriction of } C_4 \text{ to } D$ (satisfies $C_{\text{ext}}$ and $C_{\text{con}}$)
1. For every Boolean algebra \( B \) there is a totally disconnected compact regular \( T_1 \) space \( X \) such that \( B \) is isomorphic to a subalgebra of \( \text{RegCl}(X) \) (Stone [11]).
1. For every Boolean algebra $B$ there is a totally disconnected compact regular $T_1$ space $X$ such that $B$ is isomorphic to a subalgebra of $\text{RegCl}(X)$ (Stone [11]).

2. For every Boolean contact algebra which satisfies $C_{\text{ext}}$ and $C_{\text{con}}$ there is a connected compact weakly regular $T_1$ space $X$ such that $B$ is isomorphic to a subalgebra of $\langle \text{RegCl}(X), \mathcal{C}_w \rangle$ (Dimov and Vakarelov [1], Düntsch and Winter [5]).
References

Representation theorems

1. For every Boolean algebra $B$ there is a totally disconnected compact regular $T_1$ space $X$ such that $B$ is isomorphic to a subalgebra of $\text{RegCl}(X)$ (Stone [11]).

2. For every Boolean contact algebra which satisfies $C_{\text{ext}}$ and $C_{\text{con}}$ there is a connected compact weakly regular $T_1$ space $X$ such that $B$ is isomorphic to a subalgebra of $\langle \text{RegCl}(X), C_w \rangle$ (Dimov and Vakarelov [1], Düntsch and Winter [5]).

3. For every atomless Boolean algebra $B$ there is a connected compact weakly regular $T_1$ space $X$ such that $B$ is isomorphic to a subalgebra of $\text{RegCl}(X)$.
Mnogo blagodarya
Thank you
Dziękuję
Danke
Merci


