# $n$-REA Degrees and Traceability

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July 2nd 2011
Computability in Europe 2011, Sofia

### Definition

$\mathbf{b}$ is a strong minimal cover of $\mathbf{a}$ if $\mathcal{D}(< \mathbf{b}) = \mathcal{D}(\leq \mathbf{a})$.

### Question [Spector]

Classify all the degrees which have strong minimal covers.

### Theorem [Ishmukhametov]

An r.e. degree has a strong minimal cover if and only if it is array recursive.

### Definition

A degree is $1$-REA if it is an r.e. degree; a degree is $(n+1)$-REA if it is r.e. in and above an $n$-REA degree.

### Theorem [C.]

An $n$-REA degree has a strong minimal cover if and only if it is array recursive.

Ishmukhametov's Proof:

**Definition**

A degree $\mathbf{a}$ is r.e. traceable if there is a recursive function $f(n)$ such that for every function $g(n) \leq_T \mathbf{a}$, there is a recursive function $h(n)$ such that for every $n$, $g(n) \in W_{h(n)}$ and $|W_{h(n)}| \leq f(n)$.

**Definition**

A degree $\mathbf{a}$ is array recursive if every function $f$ recursive in $\mathbf{a}$ is dominated by $m_K$, the least modulus function of $K$, i.e., $f(x) \leq m_K(x)$ for cofinitely many $x$.

**Theorem [Downey, Jockusch, Stob]**

If a degree has a strong minimal cover, then it is array recursive.

**Theorem [Ishmukhametov]**

Every r.e. traceable degree has a strong minimal cover.

**Theorem [Ishmukhametov]**

An r.e. degree is r.e. traceable if and only if it is array recursive.

## Definition (lowness notions)

Given a notion of a class of sets $\mathcal{C}$ that can be relativized (e.g. ML-random sets), we say a set $A$ is low for $\mathcal{C}$ (Low($\mathcal{C}$)) if $\mathcal{C} = \mathcal{C}^A$, and $A$ is low for $\mathcal{C}, \mathcal{D}$ (Low($\mathcal{C}, \mathcal{D}$)) if $\mathcal{C} \subset \mathcal{D}^A$.

## Traceability and lowness notions in randomness

- Low(MLR,SR) $\iff$ r.e. traceable
- Low(CR,SR) $\iff$ Low(SR) $\iff$ computably traceable
- Low(MLR,CR) $\iff$ Low(MLR) $\iff$ Low(W2R,MLR) $\iff$ low for $K$ $\iff$ $K$-trivial $\iff$ ??

### Definition

An order function is a nondecreasing unbounded recursive function.

### Definition

A degree $\mathbf{a}$ is strongly jump traceable if for each order function $r(n)$ there is a trace $W_{h(n)}$ with bound $r(n)$ that traces the jump function of $A$, i.e., $\varphi_n^A(n)$.

### Theorem [Figueira, Nies, Stephan]

An r.e. degree is strongly jump traceable if and only if it is strongly superlow, i.e., $A'$ is $\omega$-r.e. and the "number of changes" in the approximation can be chosen to be bounded by any order function.

### Theorem [C.]

Both results generalize to the $n$-REA degrees.

Theorem [Arslanov's Completeness Criterion]

An FPF r.e. degree is complete ($\geq \mathbf{0}'$).

Theorem [Jockusch, Lerman, Soare, Solovay]

Every FPF $n$-REA degree computes $\mathbf{0}'$.

Thank you!