Quantitative Analysis of Randomized Distributed Systems and Probabilistic Automata

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joint work with Nathalie Bertrand
Frank Ciesinski
Marcus Größer

Probability elsewhere

- randomized algorithms [Rabin 1960]
  breaking symmetry, fingerprints, input sampling, ...

- stochastic control theory [Bellman 1957]
  operations research

- performance modeling [Markov, Erlang, Kolm., ∼ 1900]
  emphasis on steady-state and transient measures

- biological systems, resilient systems, security protocols
  ⋮

Probability elsewhere

- randomized algorithms [Rabin 1960]
  breaking symmetry, fingerprints, input sampling, ...
  models: discrete-time Markov chains
  Markov decision processes
- stochastic control theory [Bellman 1957]
  operations research
  models: Markov decision processes
- performance modeling [Markov, Erlang, Kolm., ∼ 1900]
  emphasis on steady-state and transient measures
  models: continuous-time Markov chains
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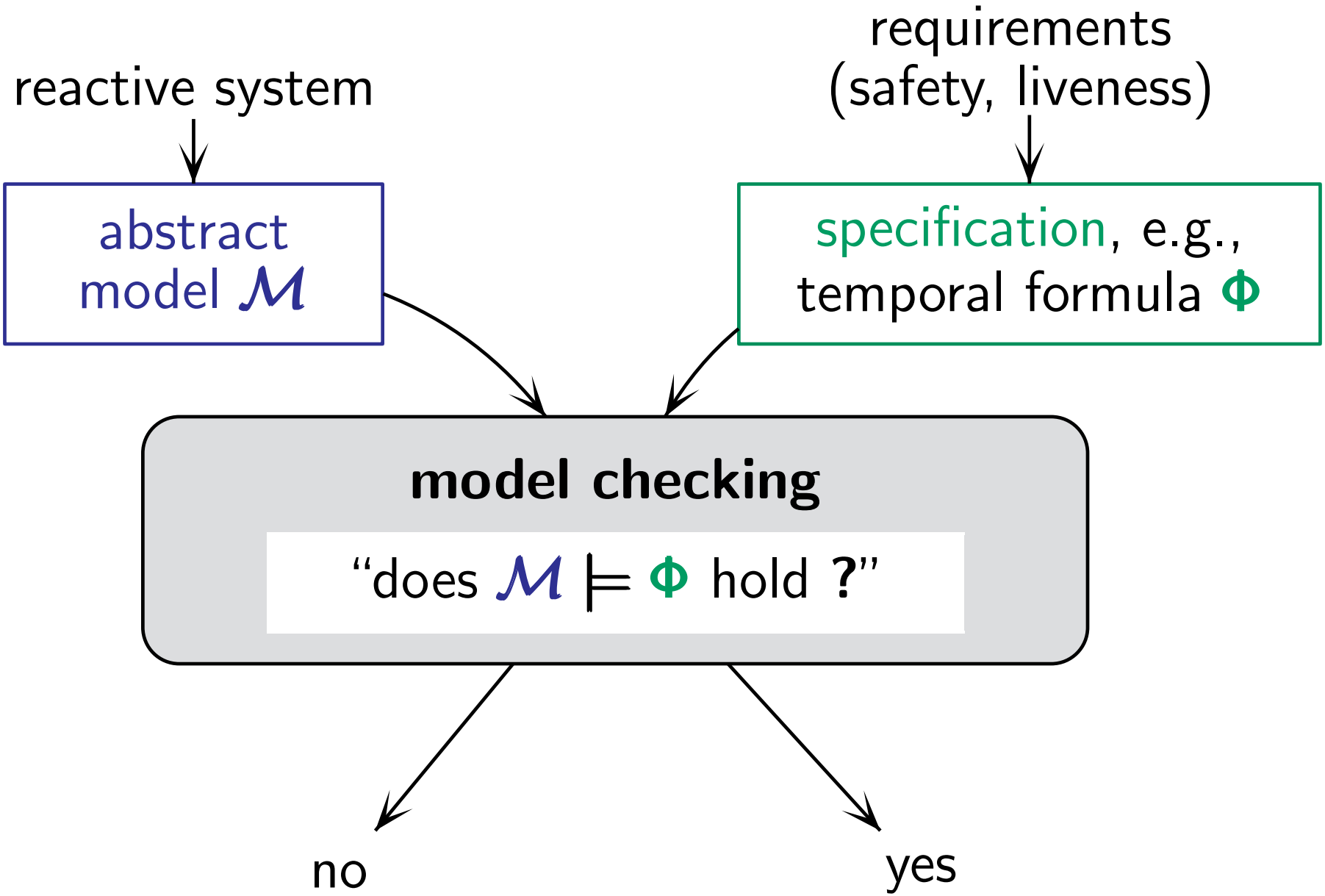
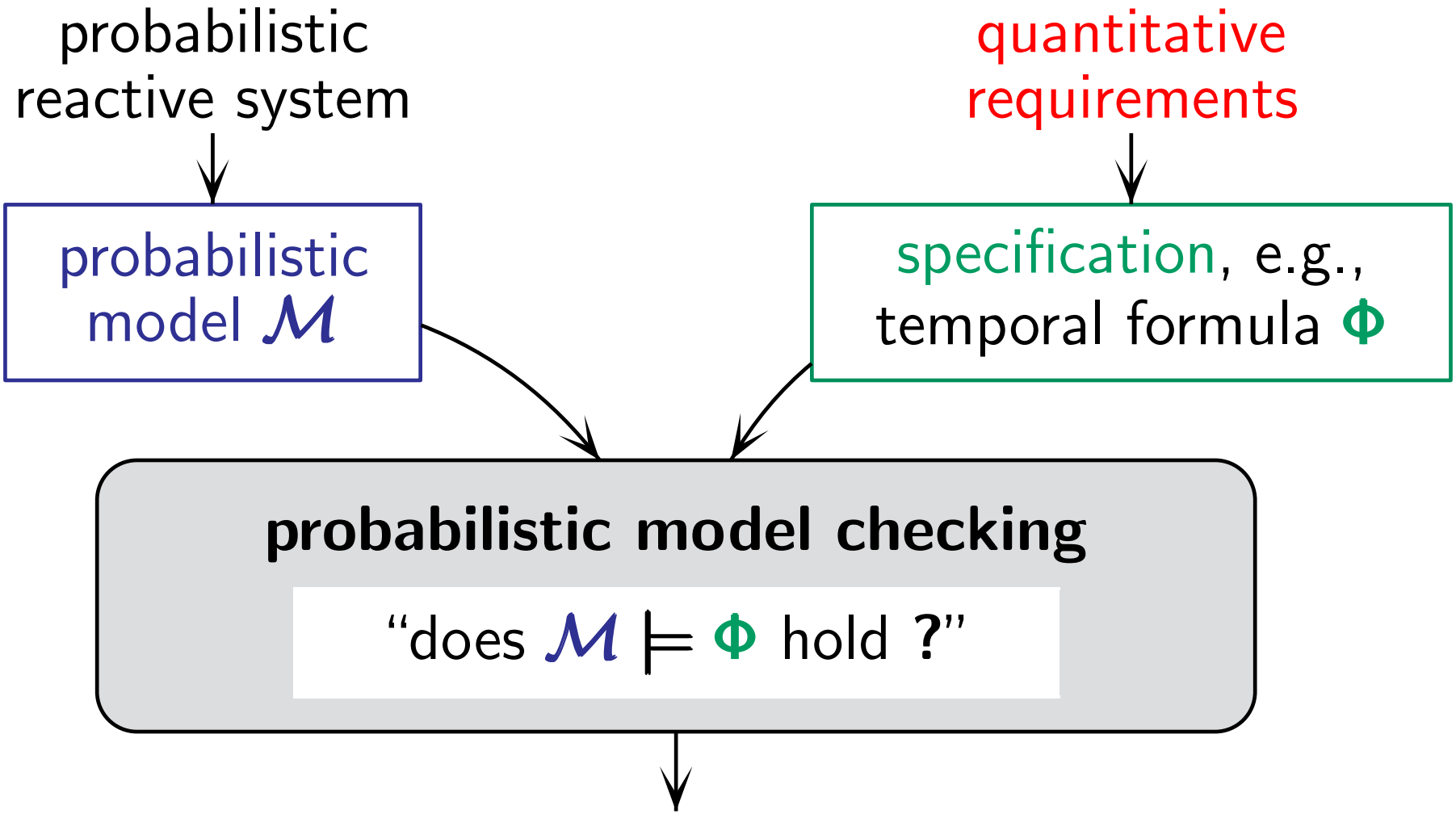
Model checking

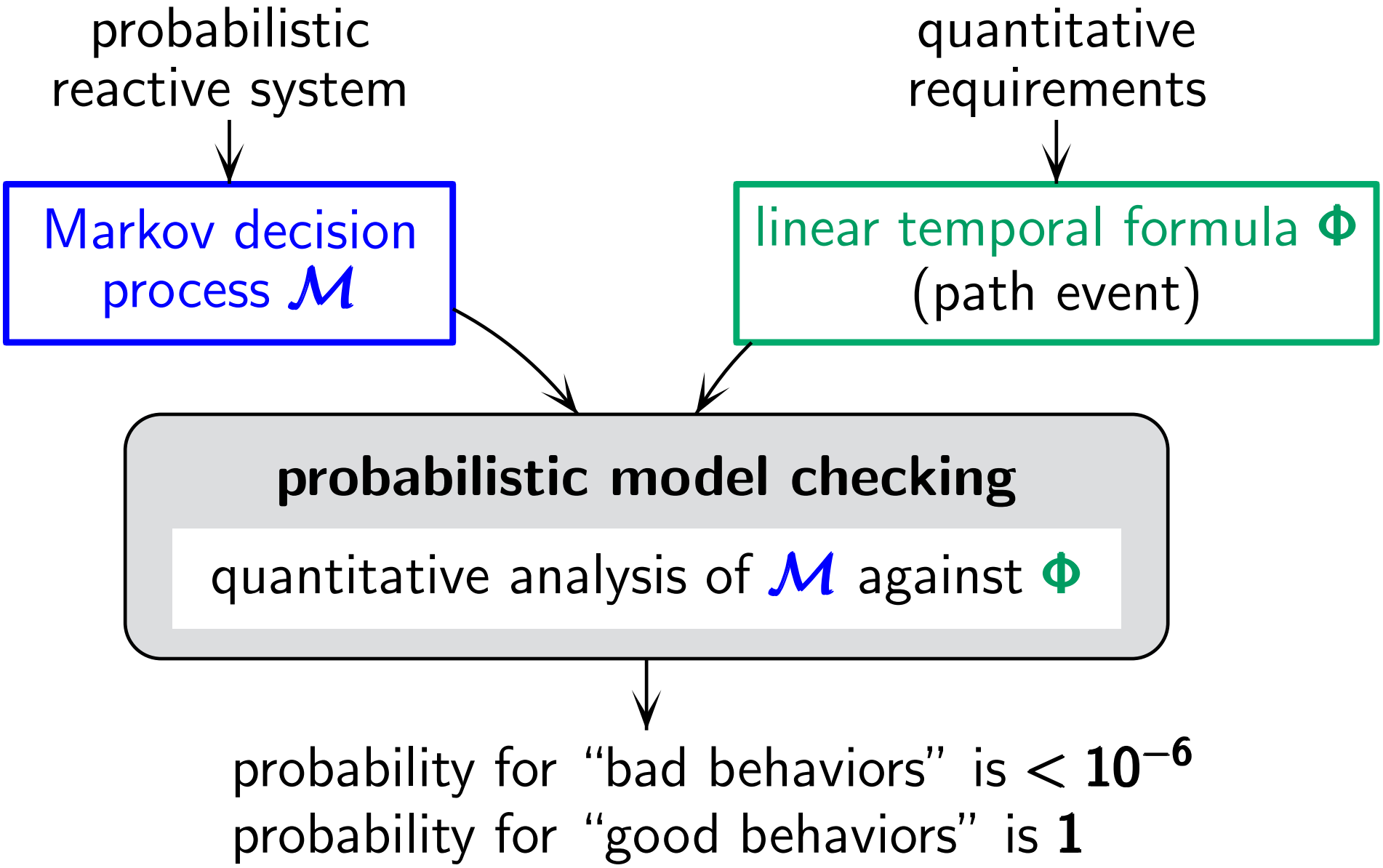


Probabilistic model checking



probability for "bad behaviors" is $< \mathbf{10^{-6}}$
probability for "good behaviors" is $\mathbf{1}$
expected costs for ....

Probabilistic model checking



Outline

- Markov decision processes (MDP) and quantitative analysis against path events
- partial order reduction for MDP
- partially-oberservable MDP
- conclusions

Markov decision process (MDP)

operational model with nondeterminism and probabilism

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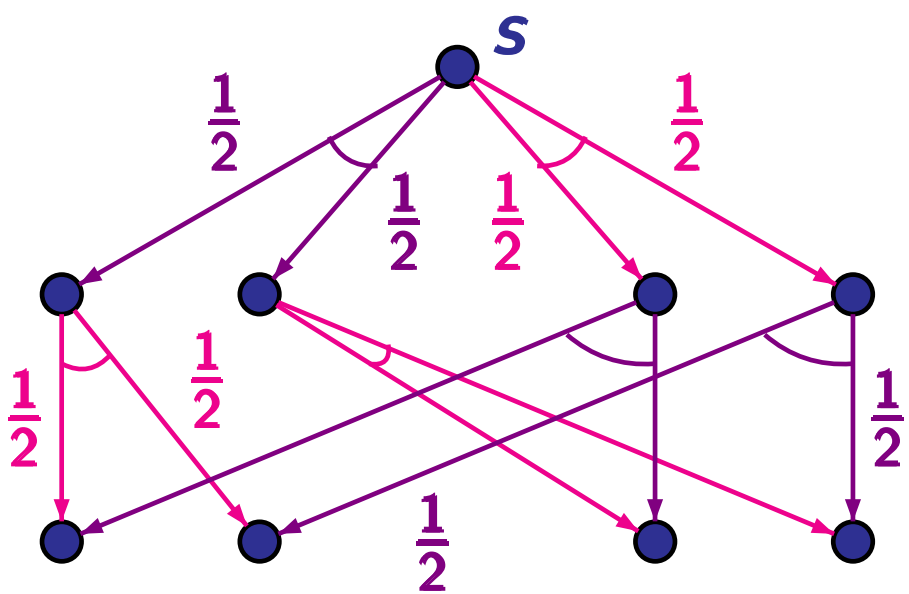
- modeling randomized distributed systems by interleaving

process 1 tosses a coin

process 2 tosses a coin



process 2 tosses a coin

process 1 tosses a coin

Markov decision process (MDP)

operational model with nondeterminism and probabilism

- modeling randomized distributed systems by interleaving
- nondeterminism useful for abstraction, underspec., modeling interactions with an unkown environment

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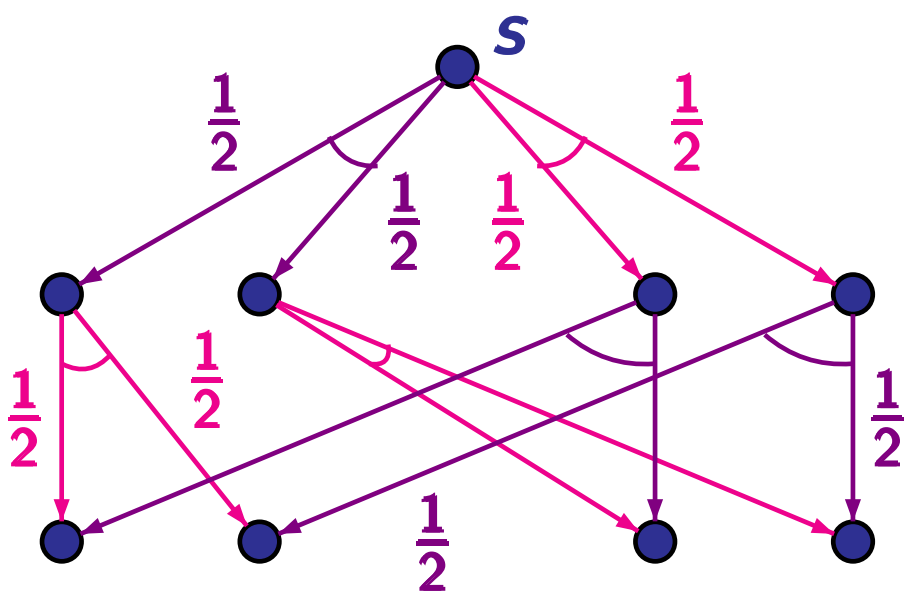

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$$\mathcal{M} = (S, Act, P, \ldots)$$

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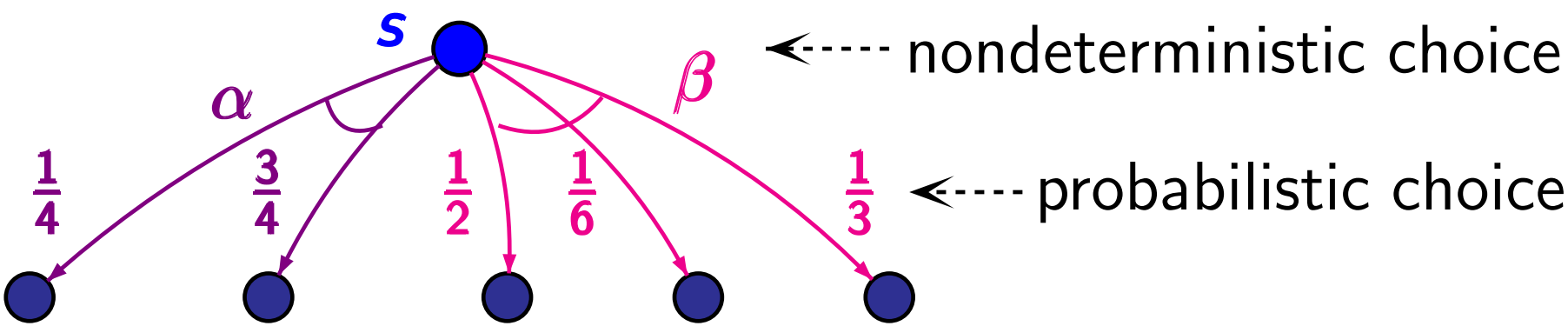
- $P : S \times Act \times S \to [0, 1]$ s.t.

$$\sum_{s' \in S} P(s, \alpha, s') = 1$$



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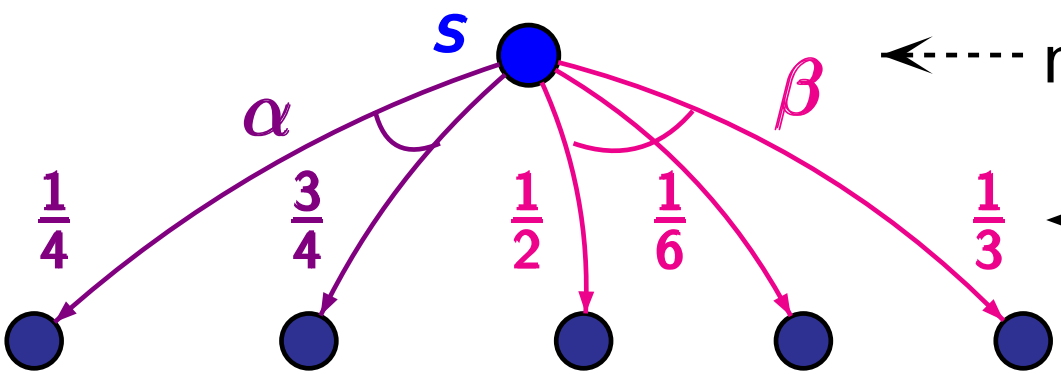
$$\forall s \in S \;\; \forall \alpha \in Act. \;\; \sum_{s' \in S} P(s, \alpha, s') \in \{0, 1\}$$

$Act(s)$ = set of actions that are enabled in state s

$\alpha \notin Act(s)$ $\qquad$ $\alpha \in Act(s)$



nondeterministic choice

probabilistic choice

Markov decision process (MDP)

$\mathcal{M} = (S, Act, P, s_0, AP, L, rew, \ldots)$

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- s_0 initial state
- AP set of atomic propositions
- labeling $L : S \to 2^{AP}$
- reward function $rew : S \times Act \to \mathbb{R}$

Randomized mutual exclusion protocol

- 2 concurrent processes $\mathcal{P}_1$, $\mathcal{P}_2$ with 3 phases:
  - n_i noncritical actions of process $\mathcal{P}_i$
  - w_i waiting phase of process $\mathcal{P}_i$
  - c_i critical section of process $\mathcal{P}_i$
- competition of both processes are waiting

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- resolved by a randomized arbiter who tosses a coin

Randomized mutual exclusion protocol

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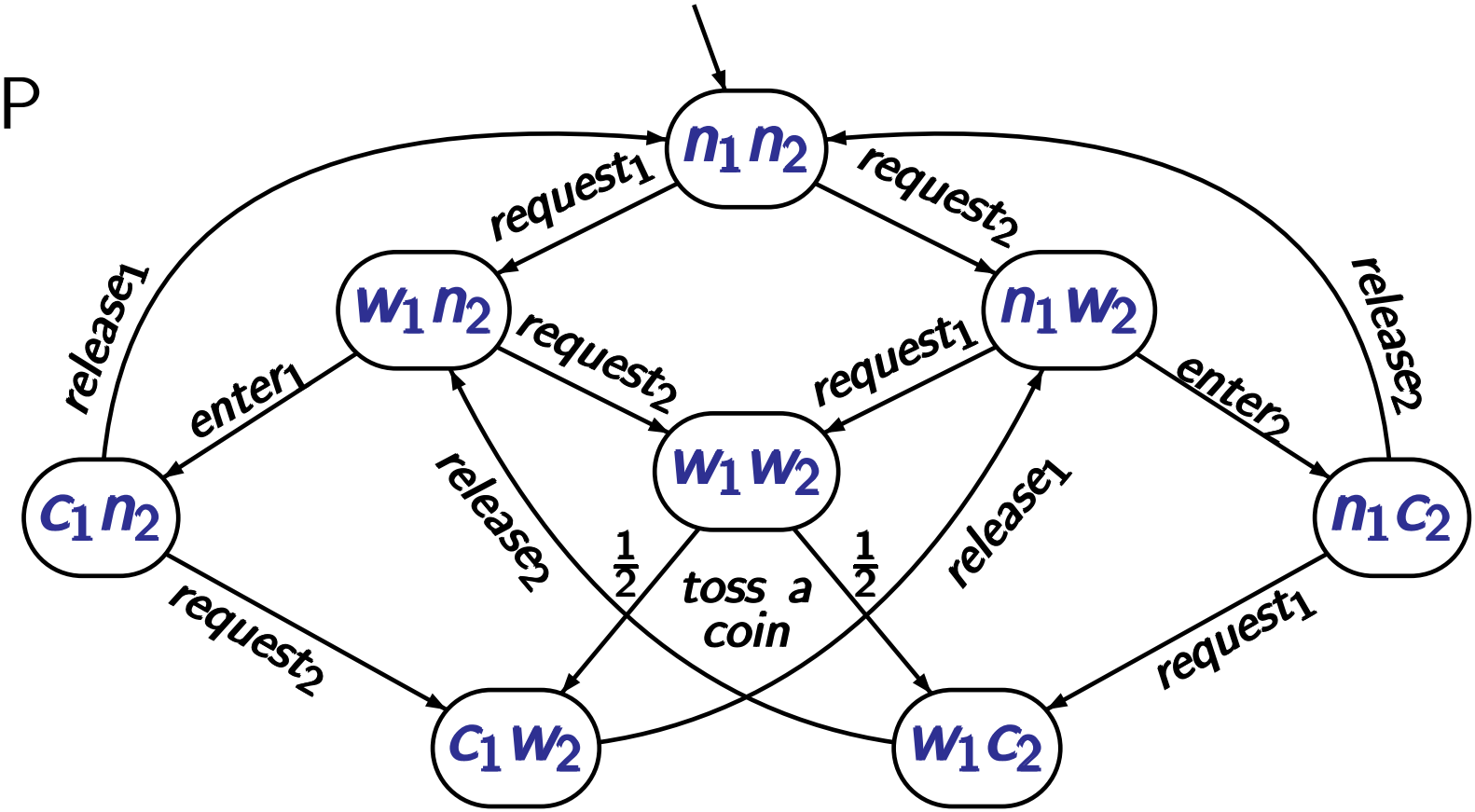

- interleaving of the request operations
- competition if both processes are waiting
- randomized arbiter tosses a coin if both are waiting

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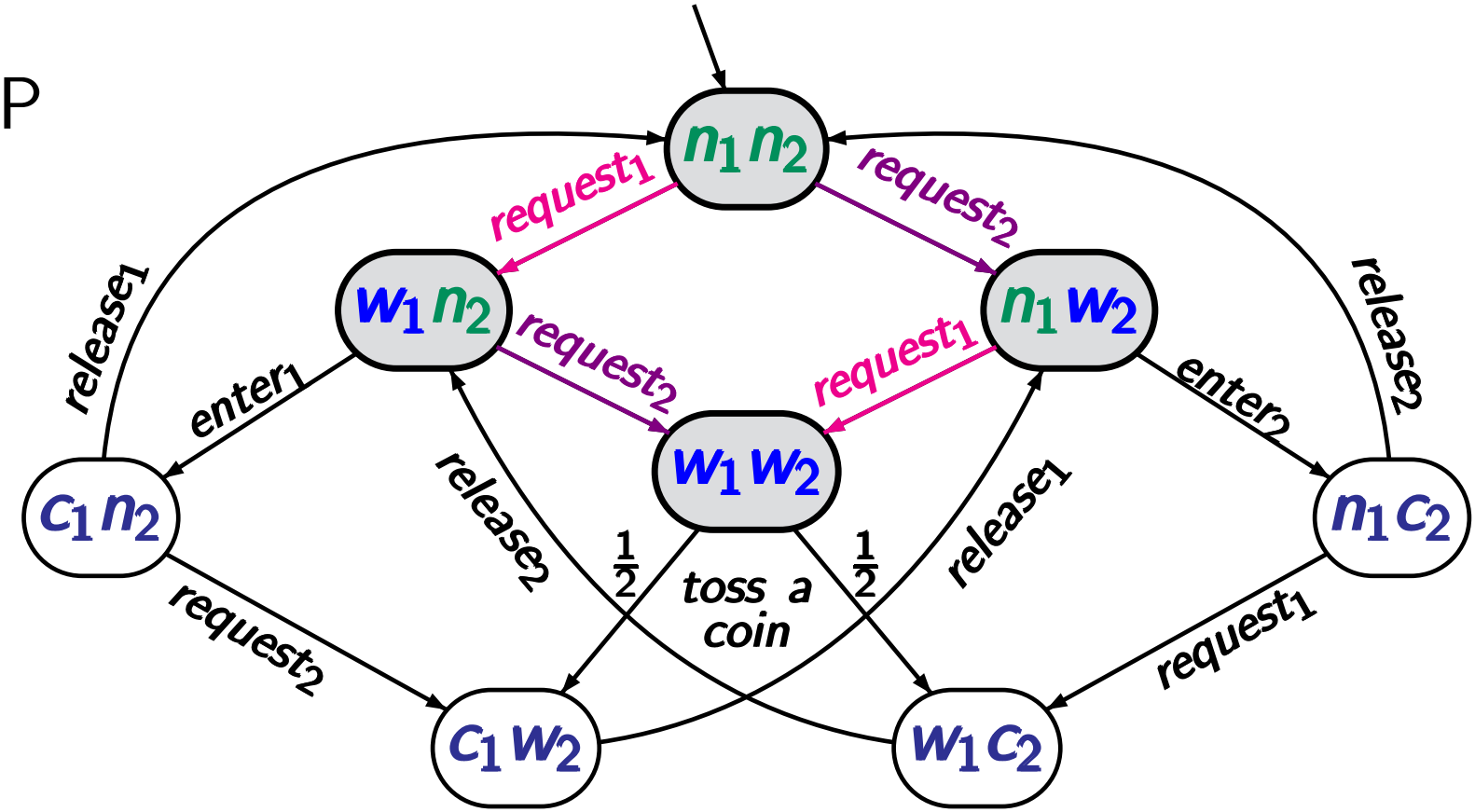

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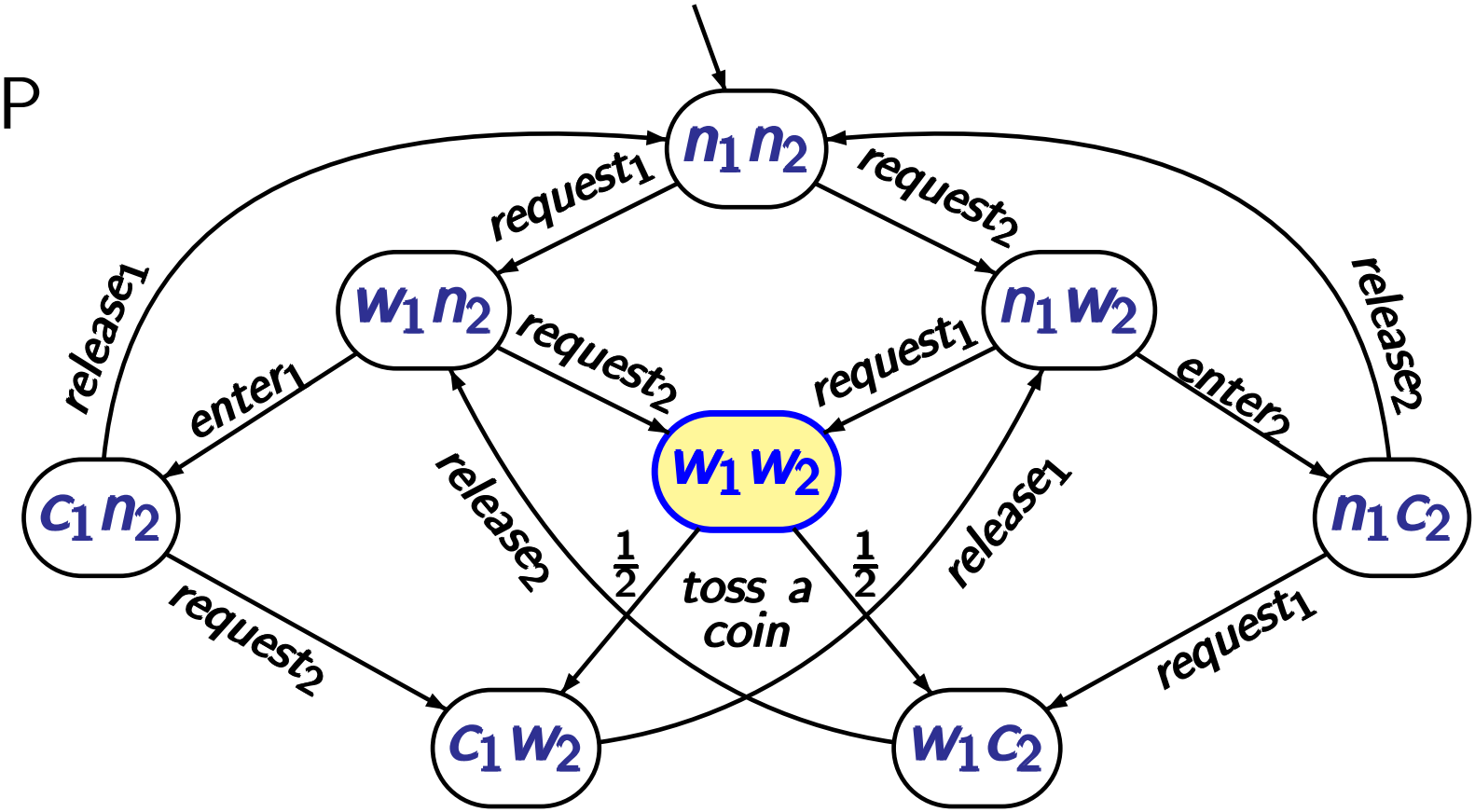

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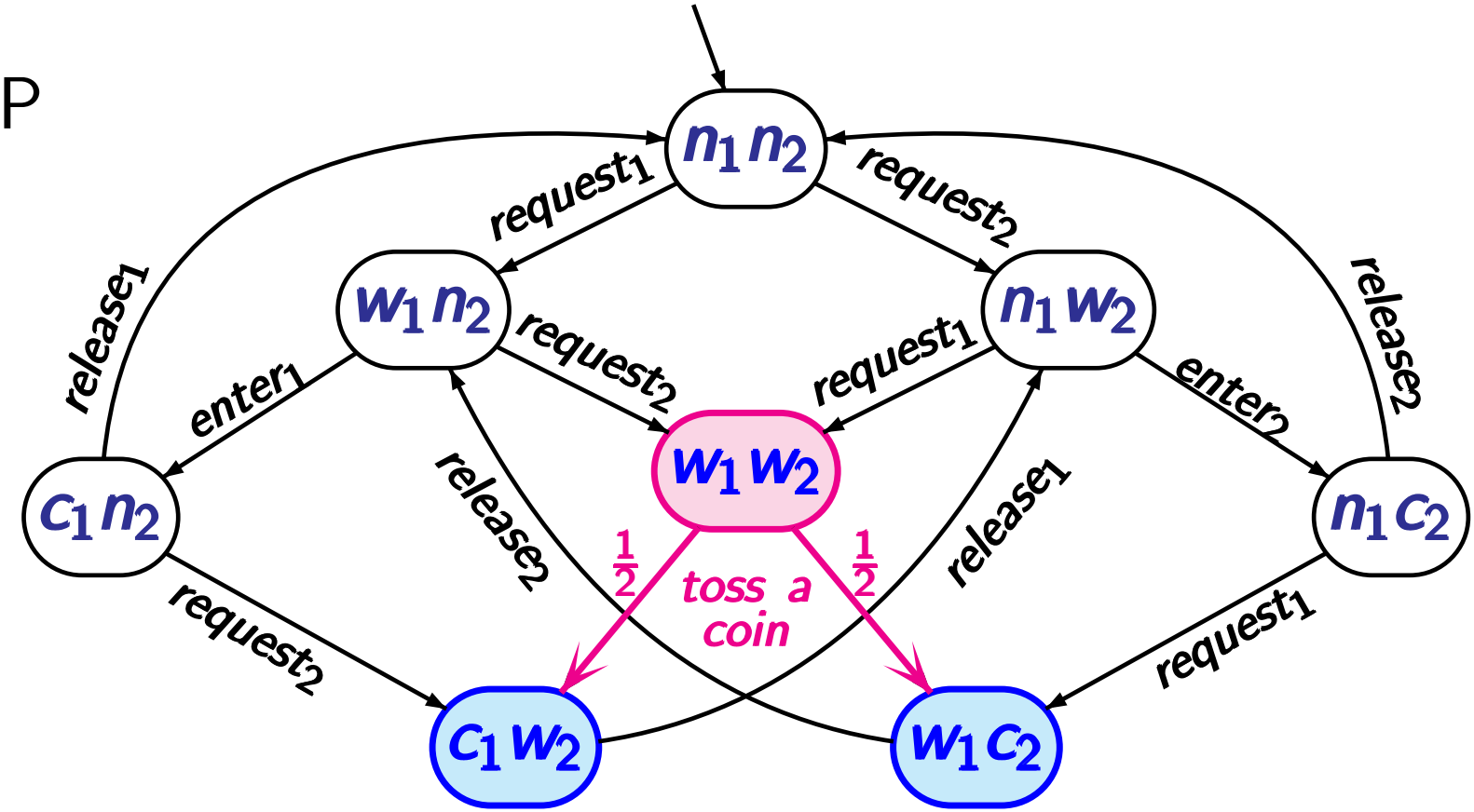

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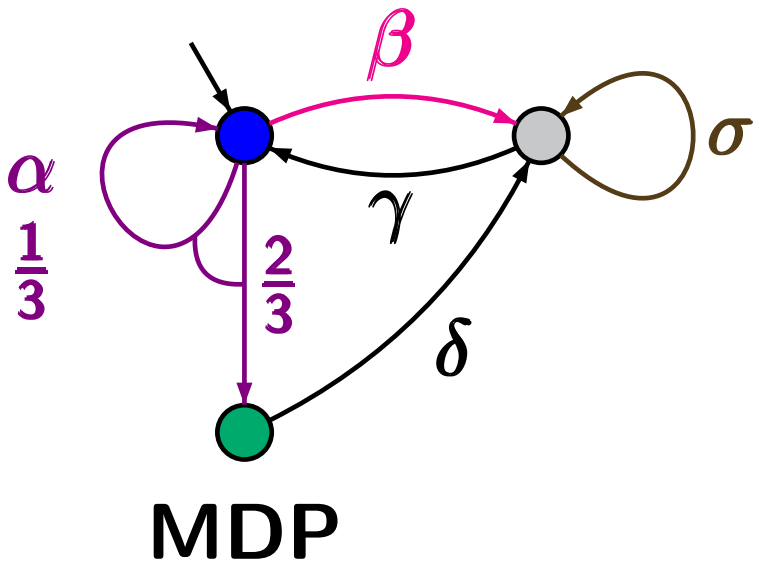

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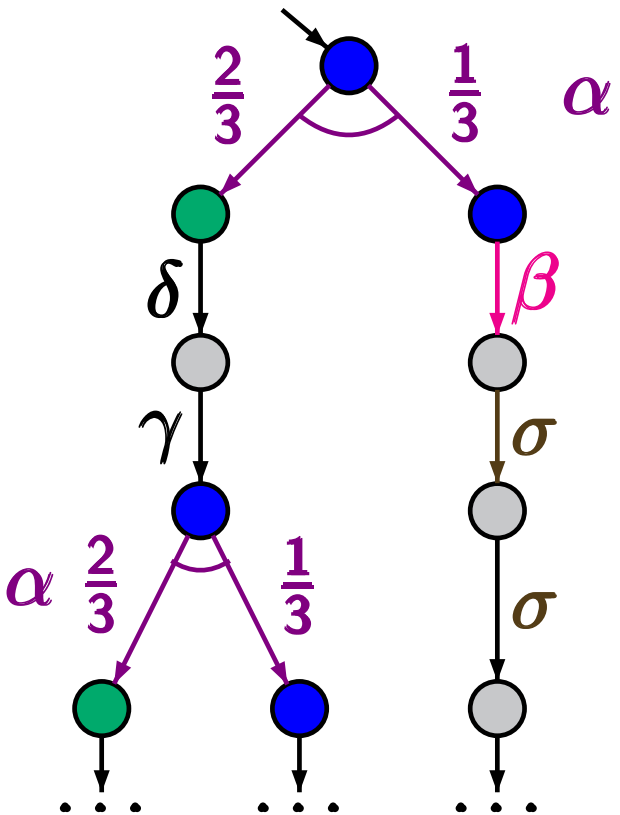

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typical task: given a measurable path event E,

* check whether E holds almost surely, i.e., $\Pr^D(E) = 1$ for all schedulers D

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yields a notion of probability measure $\Pr^D$ on measurable sets of infinite paths

typical task: given a measurable path event E,

* check whether E holds almost surely
* compute the worst-case probability for E, i.e.,

$$\sup_D \Pr^D(E) \quad \text{or} \quad \inf_D \Pr^D(E)$$

Quantitative analysis of MDP

given: MDP $\mathcal{M} = (S, Act, P, \ldots)$ with initial state s_0

ω-regular path event E,

e.g., given by an LTL formula

task: compute $\Pr^{\mathcal{M}}_{\max}(s_0, E) = \sup_D \Pr^D(s_0, E)$

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method: compute $x_s = \Pr^{\mathcal{M}}_{max}(s, E)$ for all $s \in S$

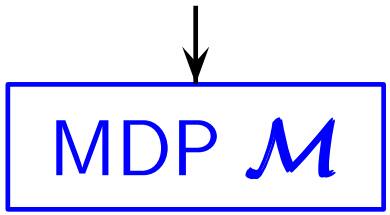
via graph analysis and linear program

[Vardi/Wolper'86]
[Courcoubetis/Yannakakis'88]
[Bianco/de Alfaro'95]
[Baier/Kwiatkowska'98]

probabilistic system

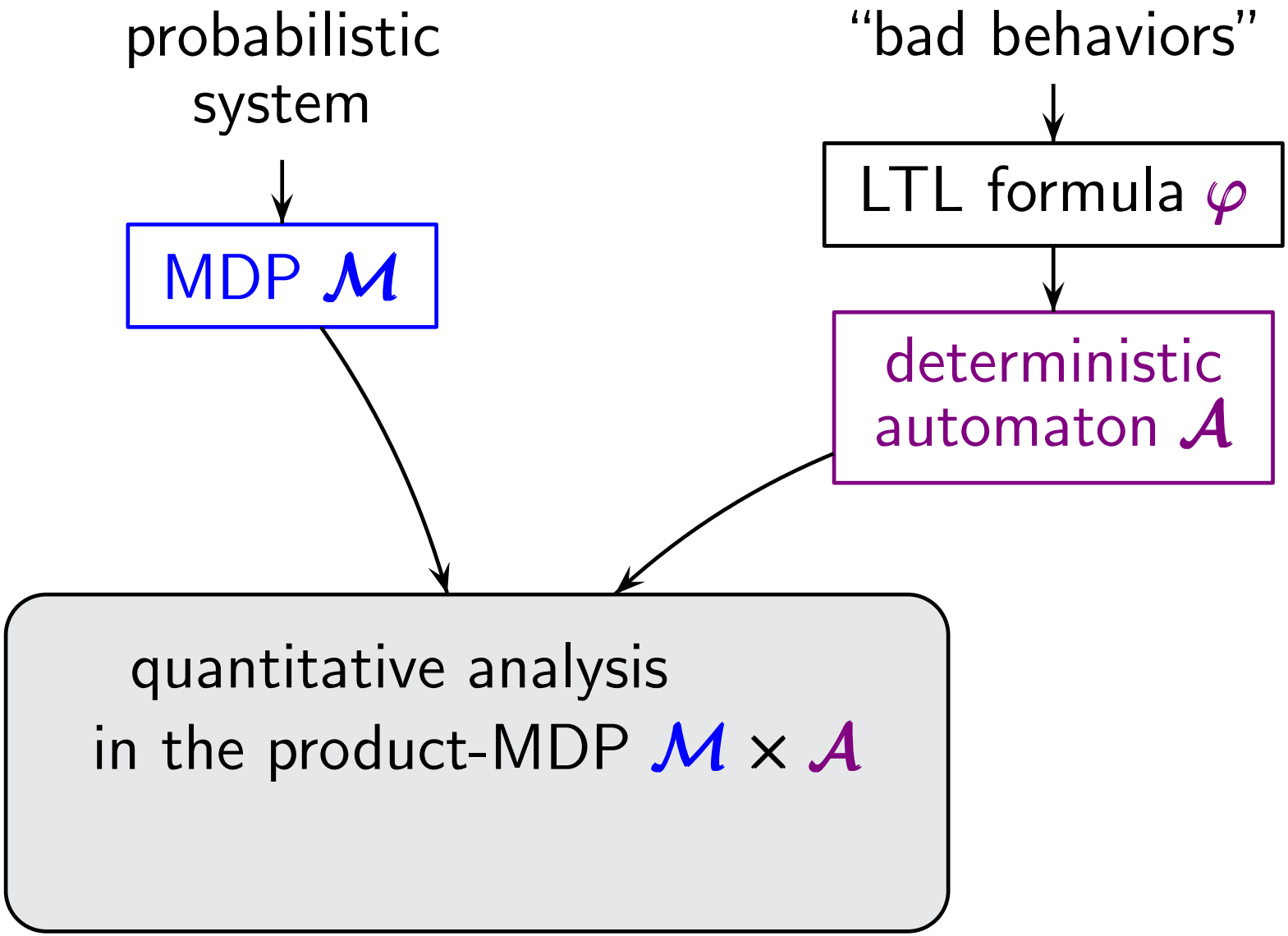
“bad behaviors”

probabilistic system

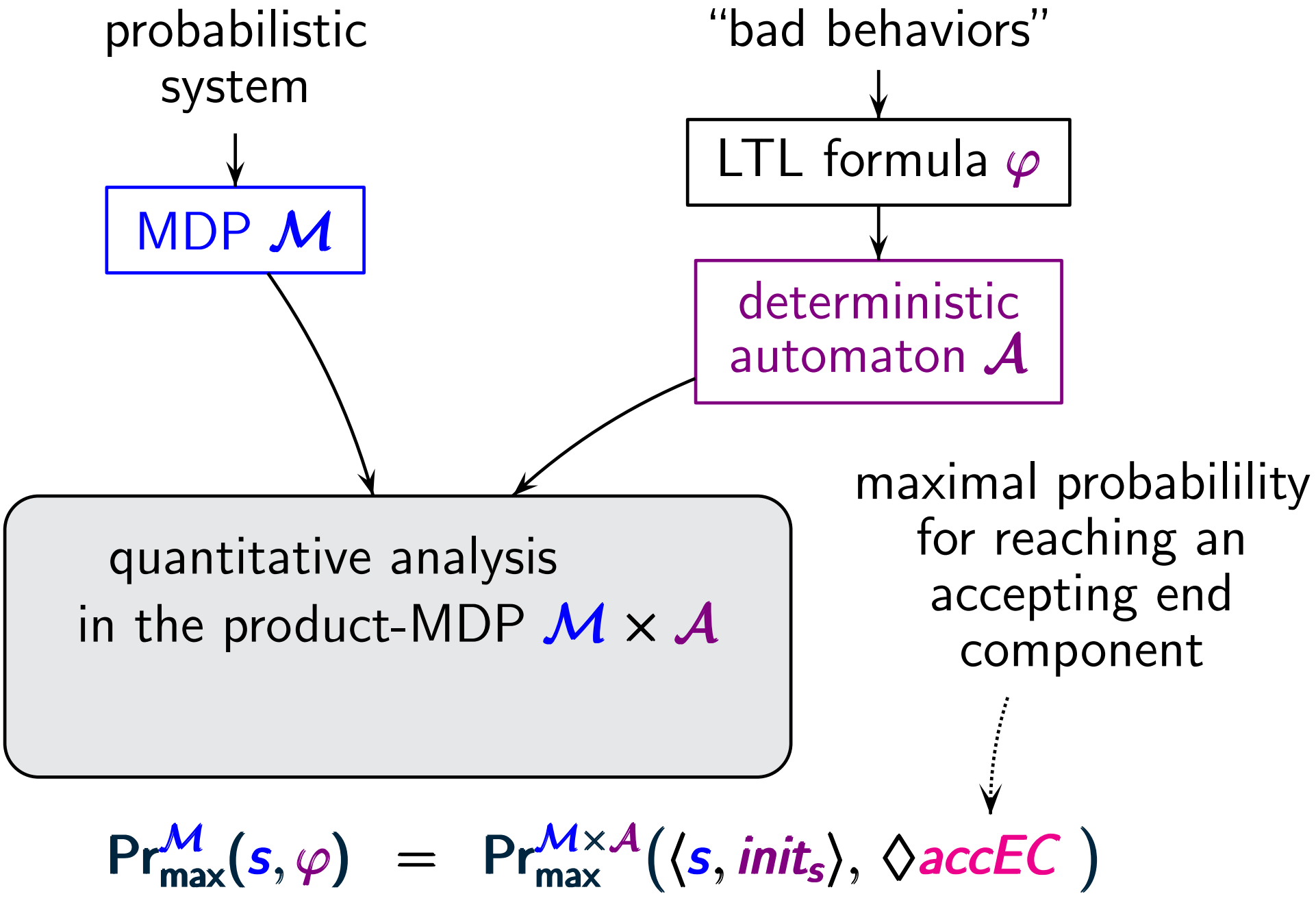


"bad behaviors"


probabilistic system
MDP $\mathcal{M}$
"bad behaviors"
LTL formula φ
deterministic automaton $\mathcal{A}$



$$\Pr^{\mathcal{M}}_{\max}(s, \varphi) \quad = \quad \Pr^{\mathcal{M}\times\mathcal{A}}_{\max}(\langle s, \mathit{init}_s \rangle, \text{ acceptance cond. of } \mathcal{A})$$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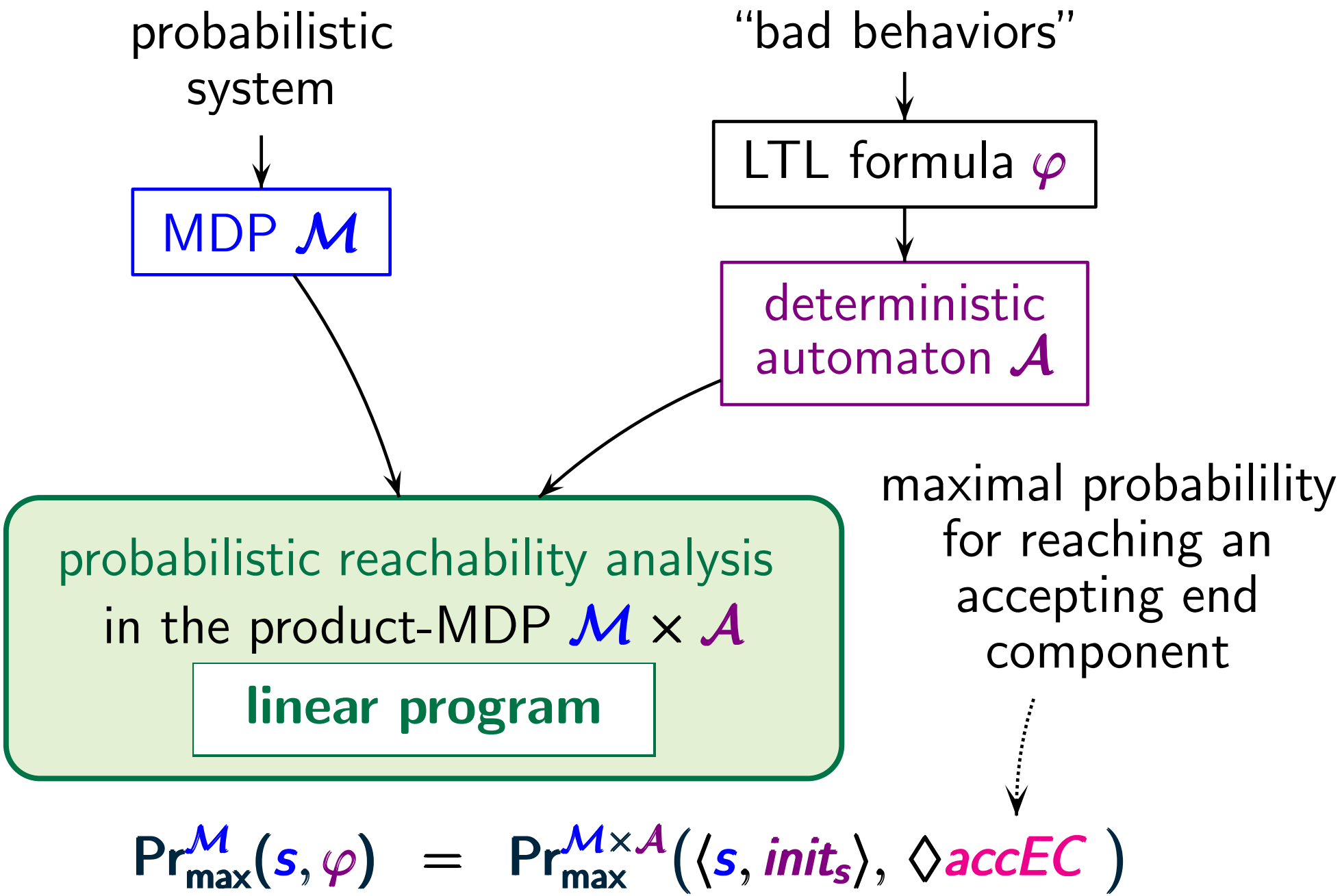
quantitative analysis in the product-MDP $\mathcal{M} \times \mathcal{A}$
maximal probabilility for reaching an accepting end component
$\Pr^{\mathcal{M}}_{\max}(s, \varphi) = \Pr^{\mathcal{M}\times\mathcal{A}}_{\max}(\langle s, init_s \rangle, \Diamond accEC)$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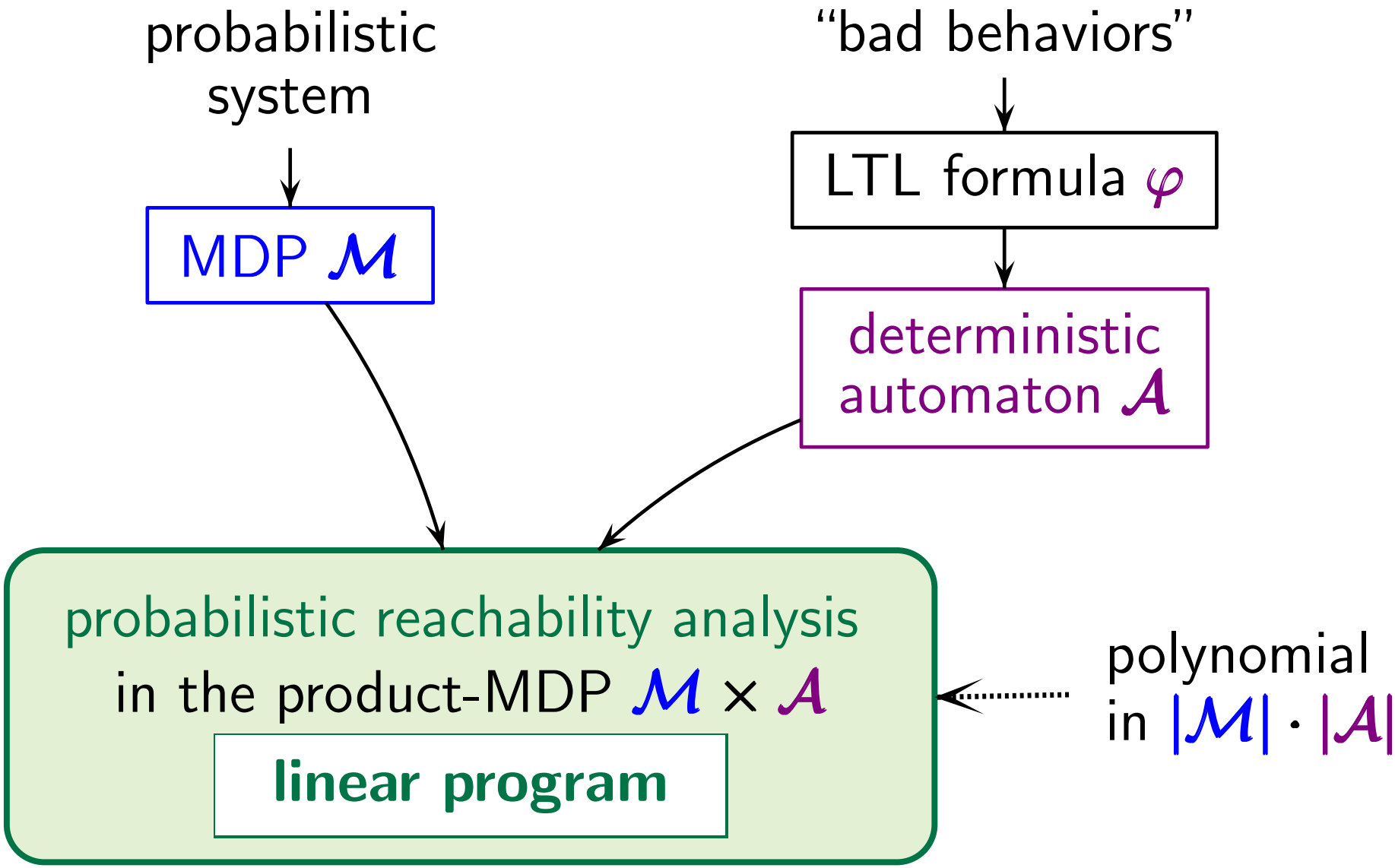

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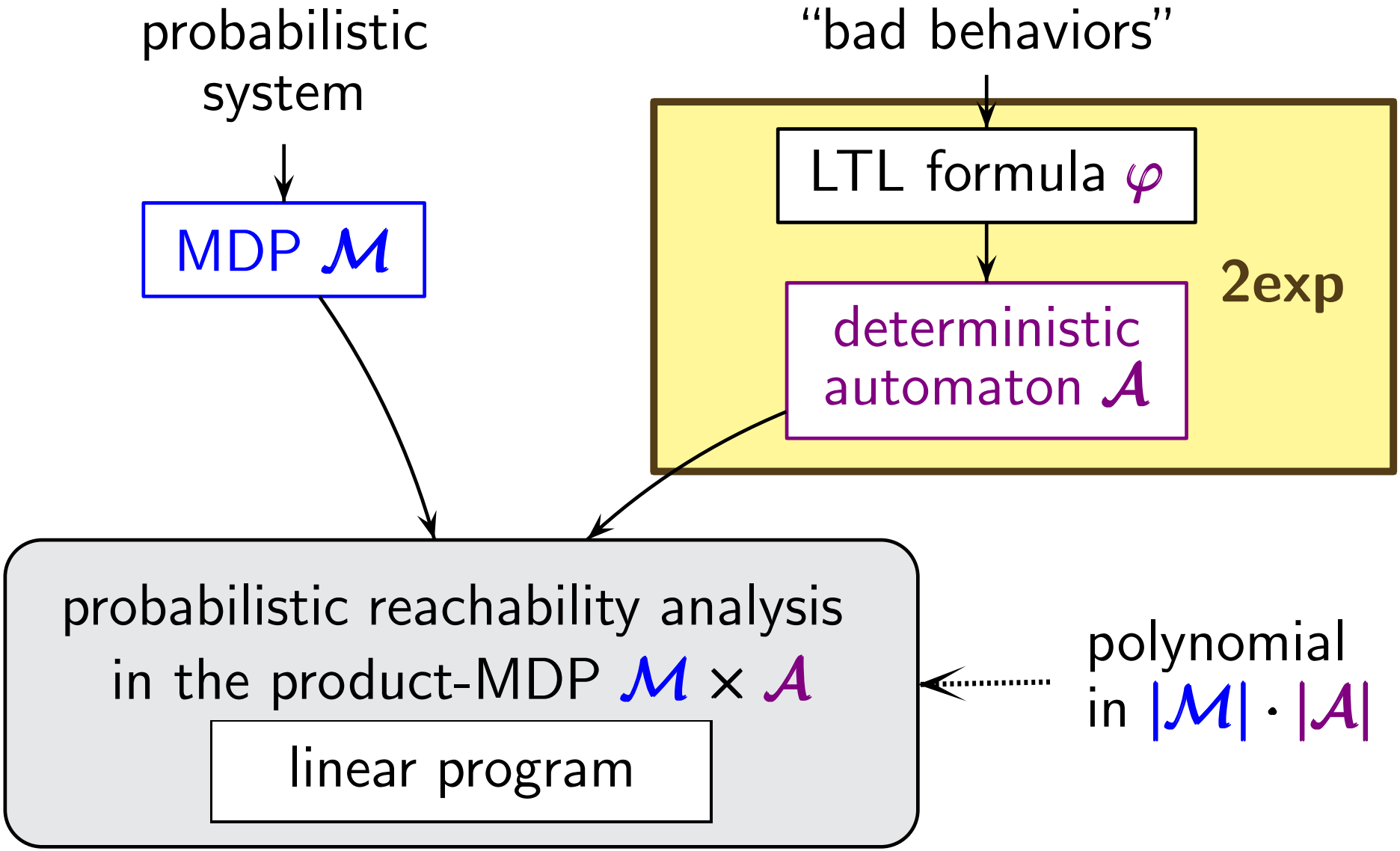

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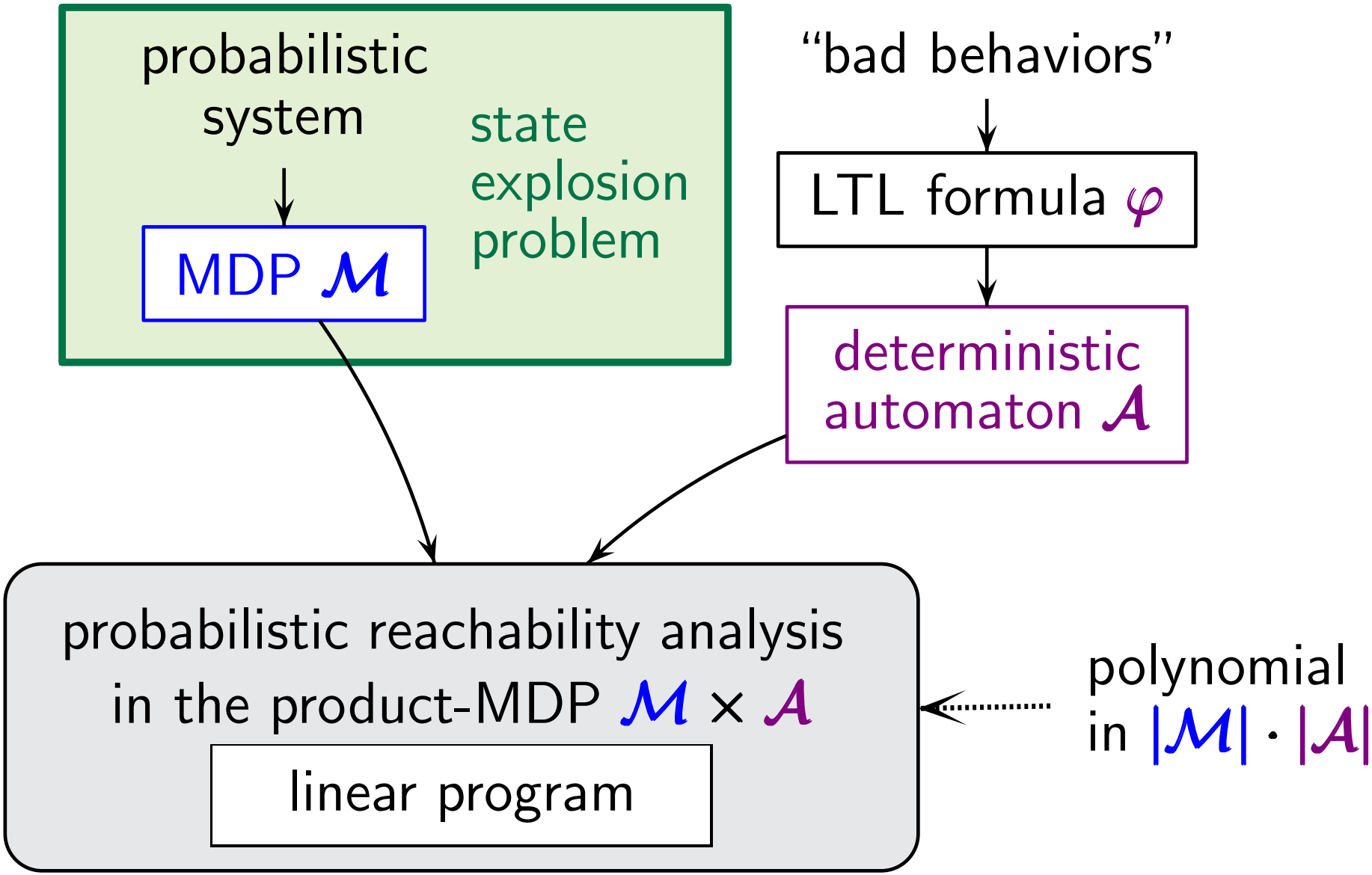

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Advanced techniques for PMC

- symbolic model checking with variants of BDDs
  e.g., in PRISM [Kwiatkowska/Norman/Parker]

- state aggregation with bisimulation
  e.g., in MRMC [Katoen et al]
- abstraction-refinement
  e.g., in RAPTURE [d'Argenio/Jeannet/Jensen/Larsen]
  PASS [Hermanns/Wachter/Zhang]
- partial order reduction
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Partial order reduction

technique for reducing the state space of concurrent systems [Godefroid,Peled,Valmari, ca. 1990]

- attempts to analyze a sub-system by identifying "redundant interleavings"
- explores representatives of paths that agree up to the order of independent actions

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e.g., $\underbrace{x := x+y}_{\text{action } \alpha} \parallel \underbrace{z := z+3}_{\text{action } \beta}$

has the same effect as $\alpha;\beta$ or $\beta;\alpha$

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DFS-based on-the-fly generation of a reduced system

for each expanded state s

- choose an appropriate subset $\mathit{Ample}(s)$ of $\mathit{Act}(s)$
- expand only the α-successors of s for $\alpha \in \mathit{Ample}(s)$ (but ignore the actions in $\mathit{Act}(s) \setminus \mathit{Ample}(s)$)

Ample-set method [Peled 1993]

given: processes $\mathcal{P}_i$ of a parallel system $\mathcal{P}_1 \| \ldots \| \mathcal{P}_n$
with transition system $\mathcal{T} = (S, Act, \rightarrow, \ldots)$

task: on-the-fly generation of a sub-system $\mathcal{T}_r$ s.t.

(A1) stutter condition ...
(A2) dependency condition ...
(A3) cycle condition ...

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$\Downarrow$

$\mathcal{T}$ and $\mathcal{T}_r$ satisfy the same stutter-invariant events, e.g., next-free LTL formulas

Ample-set method for MDP

given: processes $\mathcal{P}_i$ of a probabilistic system $\mathcal{P}_1 \| \ldots \| \mathcal{P}_n$
with MDP-semantics $\mathcal{M} = (S, Act, P, \ldots)$

task: on-the-fly generation of a sub-MDP $\mathcal{M}_r$ s.t.

$\mathcal{M}_r$ and $\mathcal{M}$ have the same extremal probabilities
for stutter-invariant events

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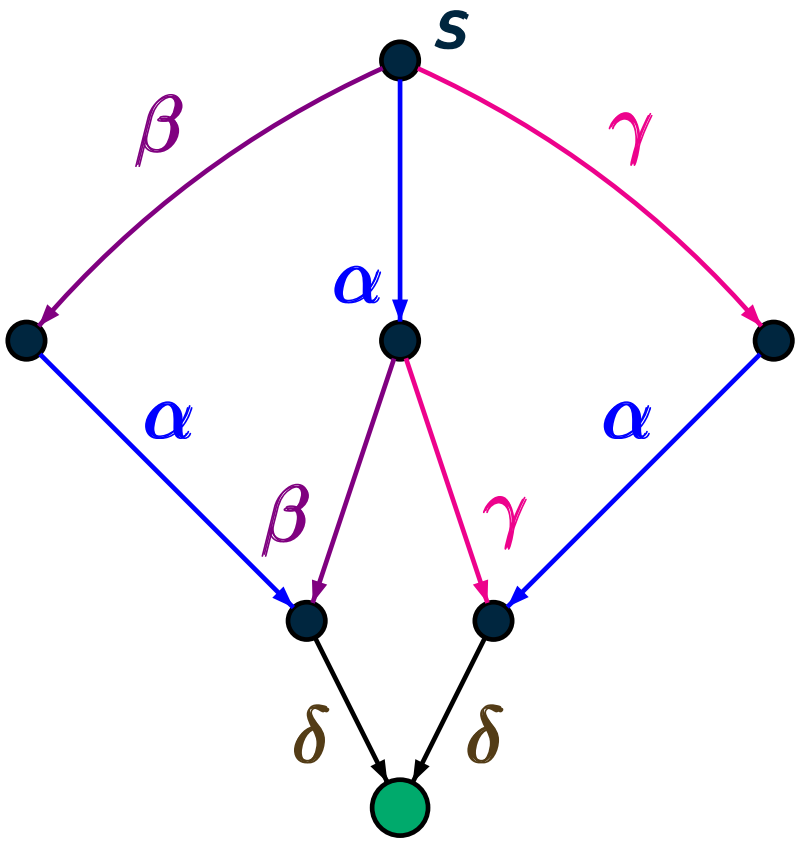
For all schedulers D for $\mathcal{M}$ there is a scheduler D_r for $\mathcal{M}_r$ s.t. for all measurable, stutter-invariant events E:

$$\Pr^{D}_{\mathcal{M}}(E) = \Pr^{D_r}_{\mathcal{M}_r}(E)$$

$$\Downarrow$$

$\mathcal{M}_r$ and $\mathcal{M}$ have the same extremal probabilities for stutter-invariant events

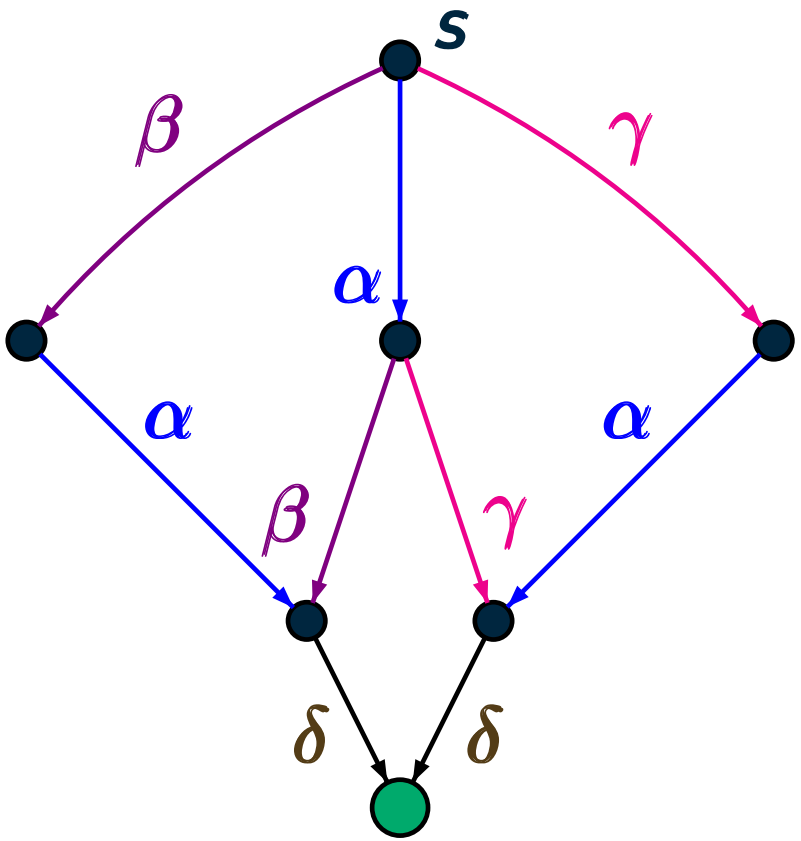
Example: ample set method



original system $\mathcal{T}$

α independent
from β and γ

Example: ample set method



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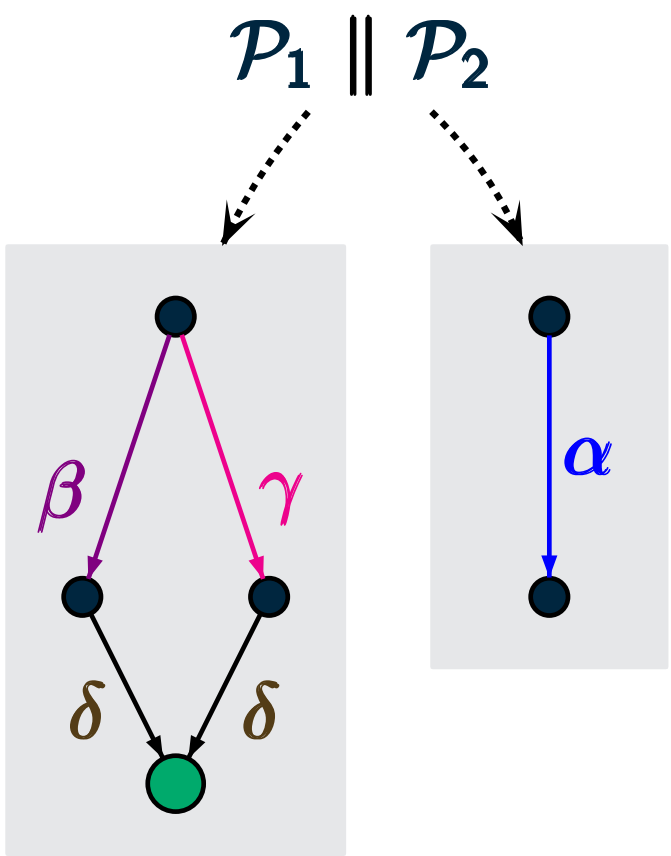

action α: $x := 1$

action δ:

IF $x > 0$ THEN $y := 1$ FI

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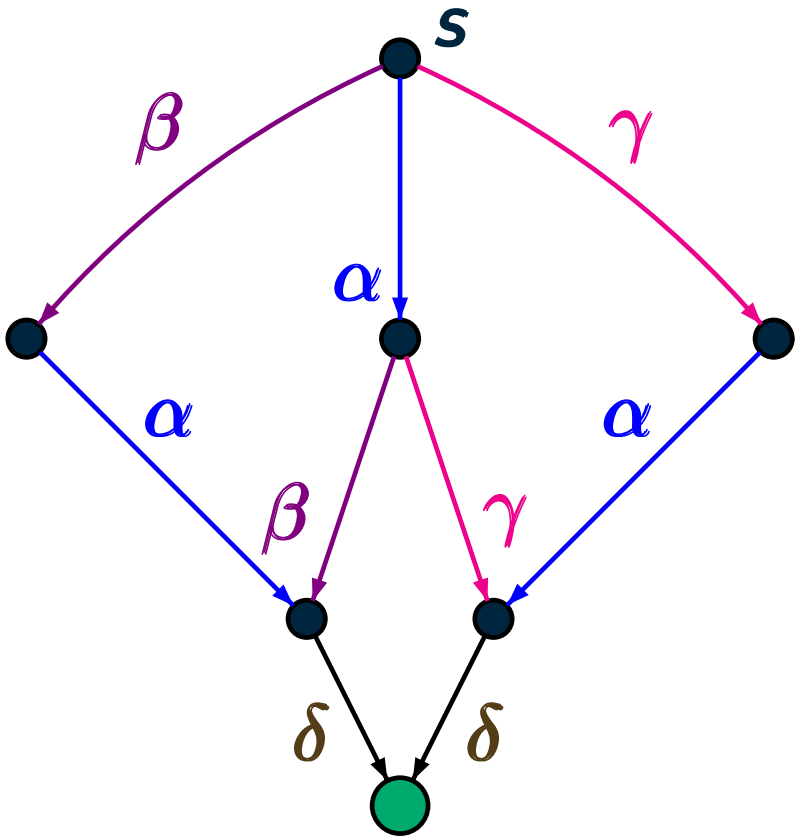

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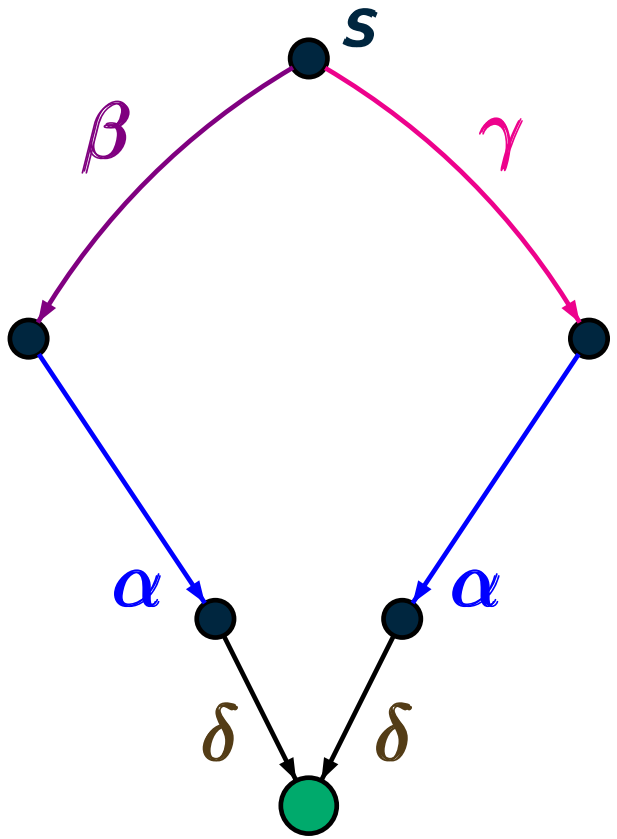
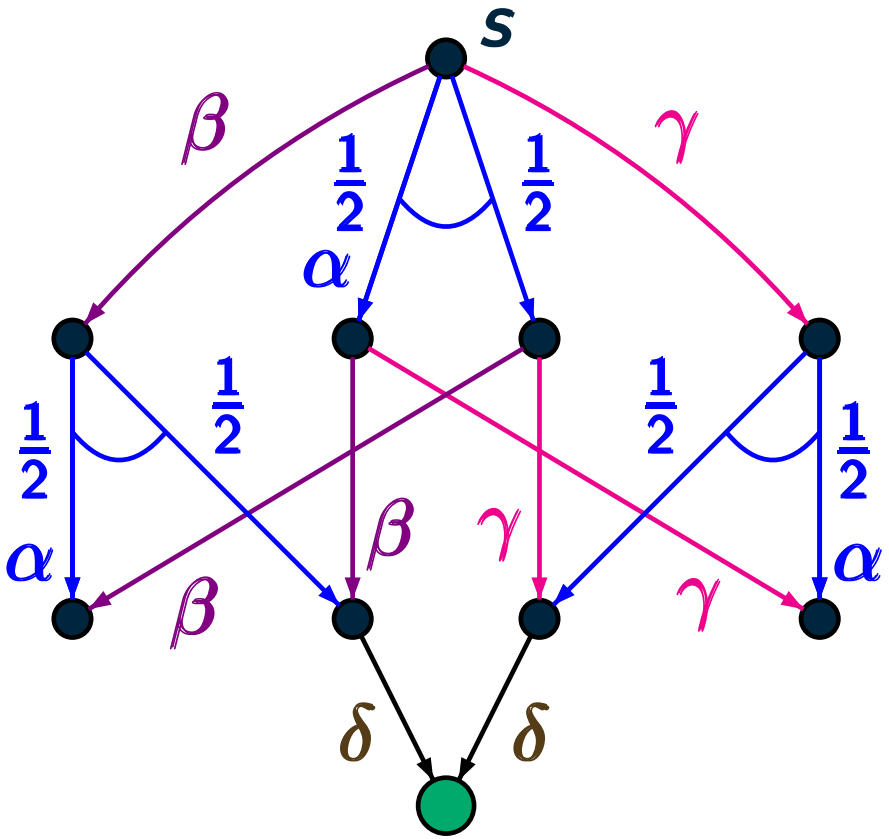


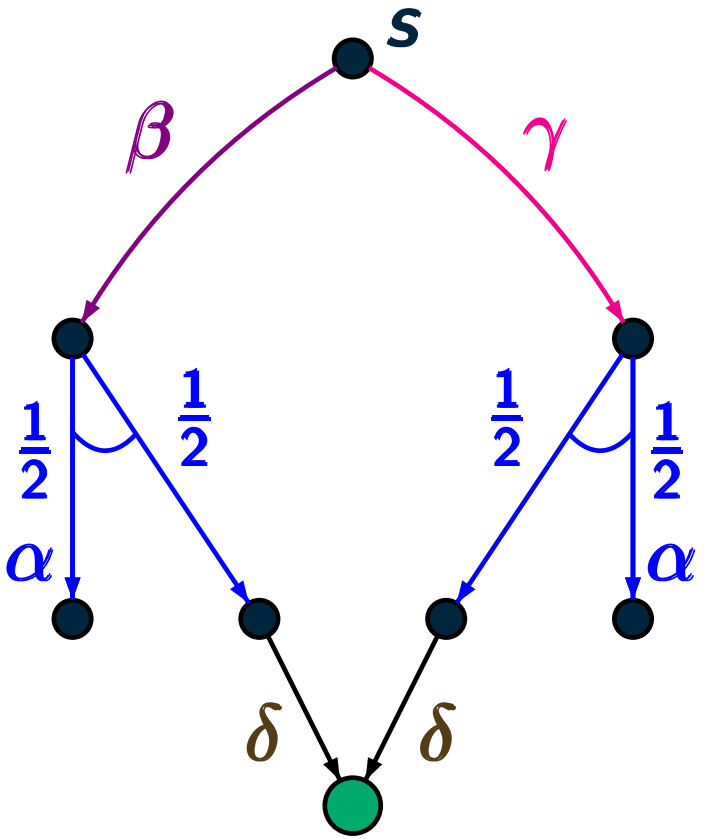
reduced system $\mathcal{T}_r$
(A1)-(A3) are fulfilled

α independent
from β and γ

Example: ample set method fails for MDP



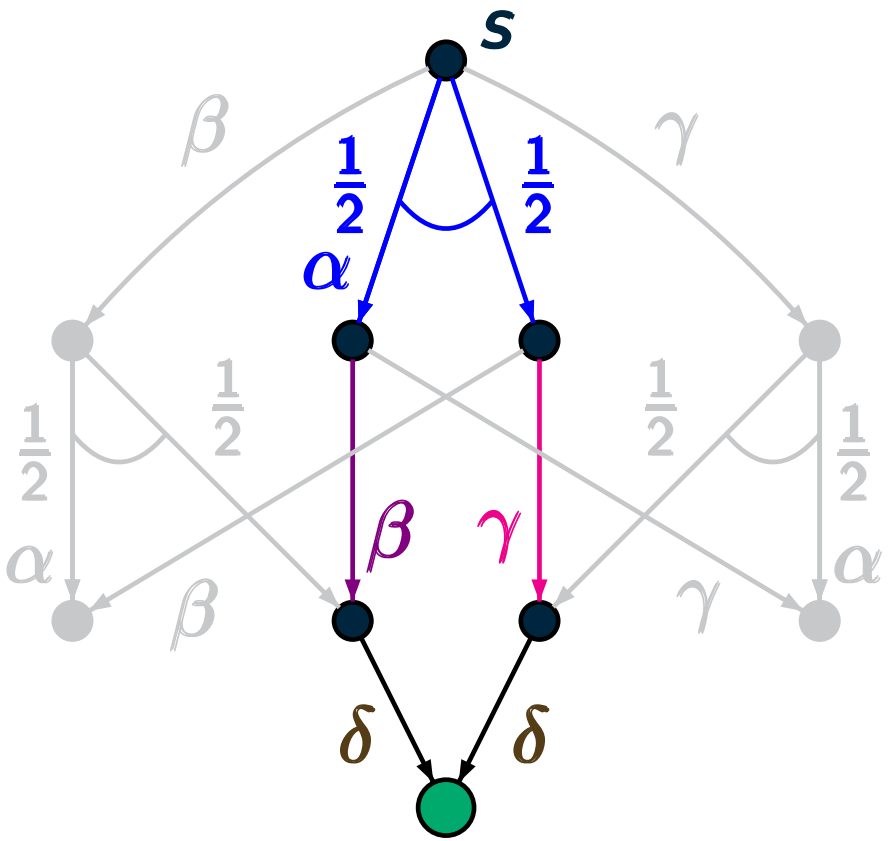
original MDP $\mathcal{M}$



reduced MDP $\mathcal{M}_r$
(A1)-(A3) are fulfilled

α independent
from β and γ

Example: ample set method fails for MDP



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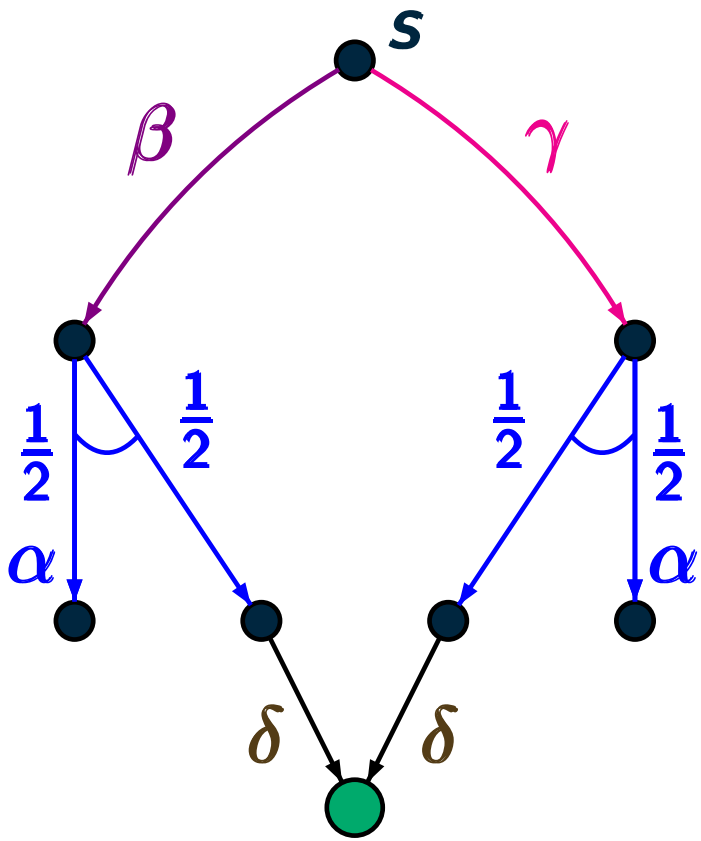
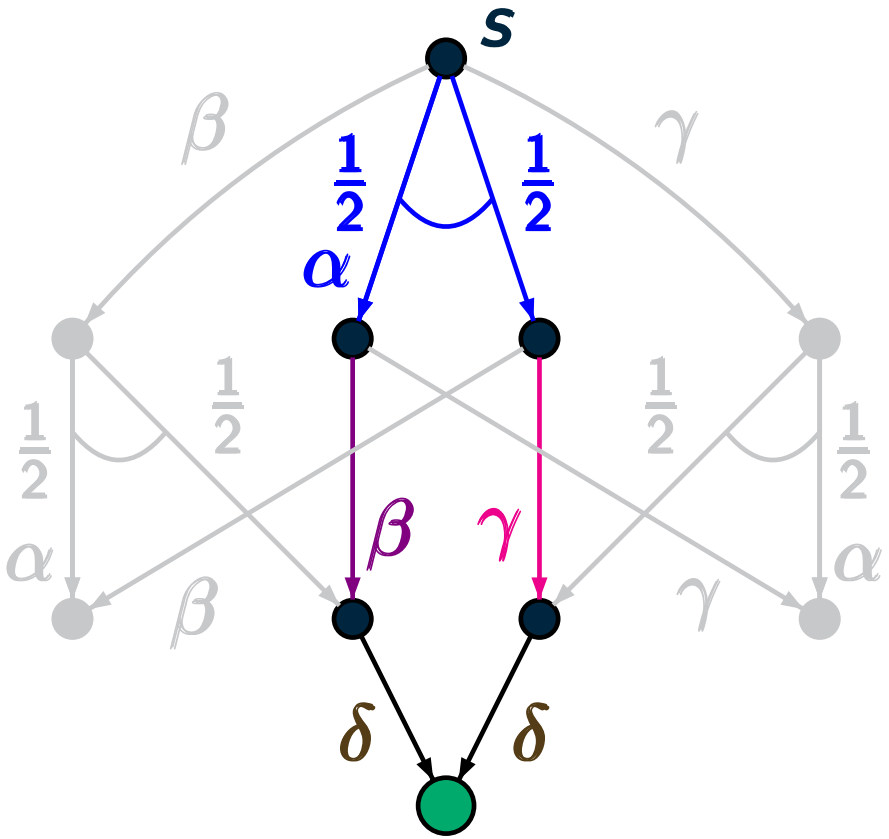


reduced MDP $\mathcal{M}_r$

$\Pr^{\mathcal{M}}_{\max}(s, \Diamond \mathit{green}) = 1$ $\Diamond$ "eventually"

Example: ample set method fails for MDP



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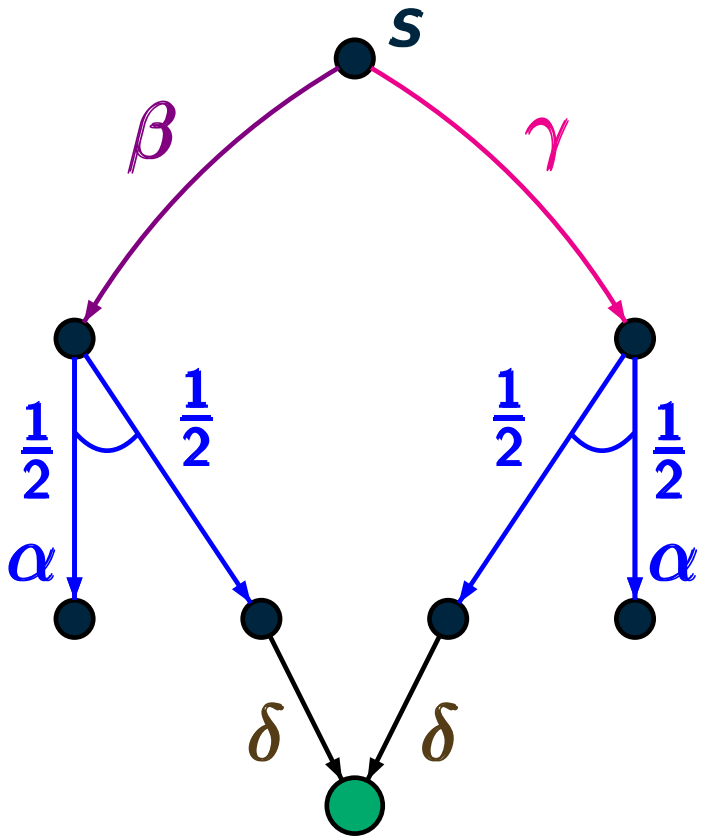

reduced MDP $\mathcal{M}_r$

$$\Pr^{\mathcal{M}}_{\max}(s, \Diamond \mathit{green}) = 1 > \frac{1}{2} = \Pr^{\mathcal{M}_r}_{\max}(s, \Diamond \mathit{green})$$

Partial order reduction for MDP

extend Peled's conditions (A1)-(A3) for the ample-sets

- (A1) stutter condition ...
- (A2) dependency condition ...
- (A3) cycle condition ...
- (A4) probabilistic condition

If there is a path $s \xrightarrow{\beta_1} \xrightarrow{\beta_2} \ldots \xrightarrow{\beta_n} \xrightarrow{\alpha}$ in $\mathcal{M}$ s.t. $\beta_1, \ldots, \beta_n, \alpha \notin \mathit{Ample}(s)$ and α is probabilistic then $|\mathit{Ample}(s)| = 1$.

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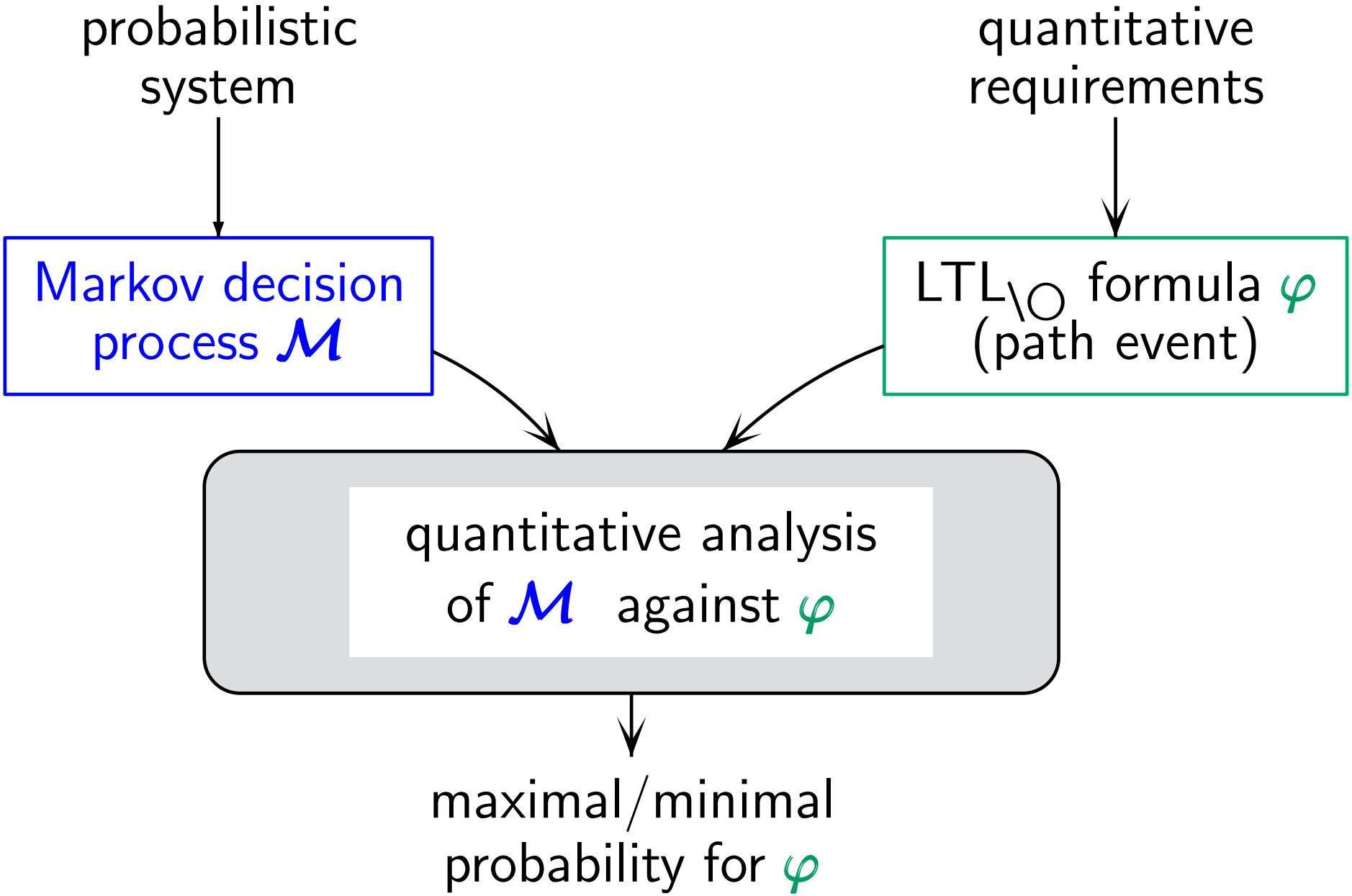
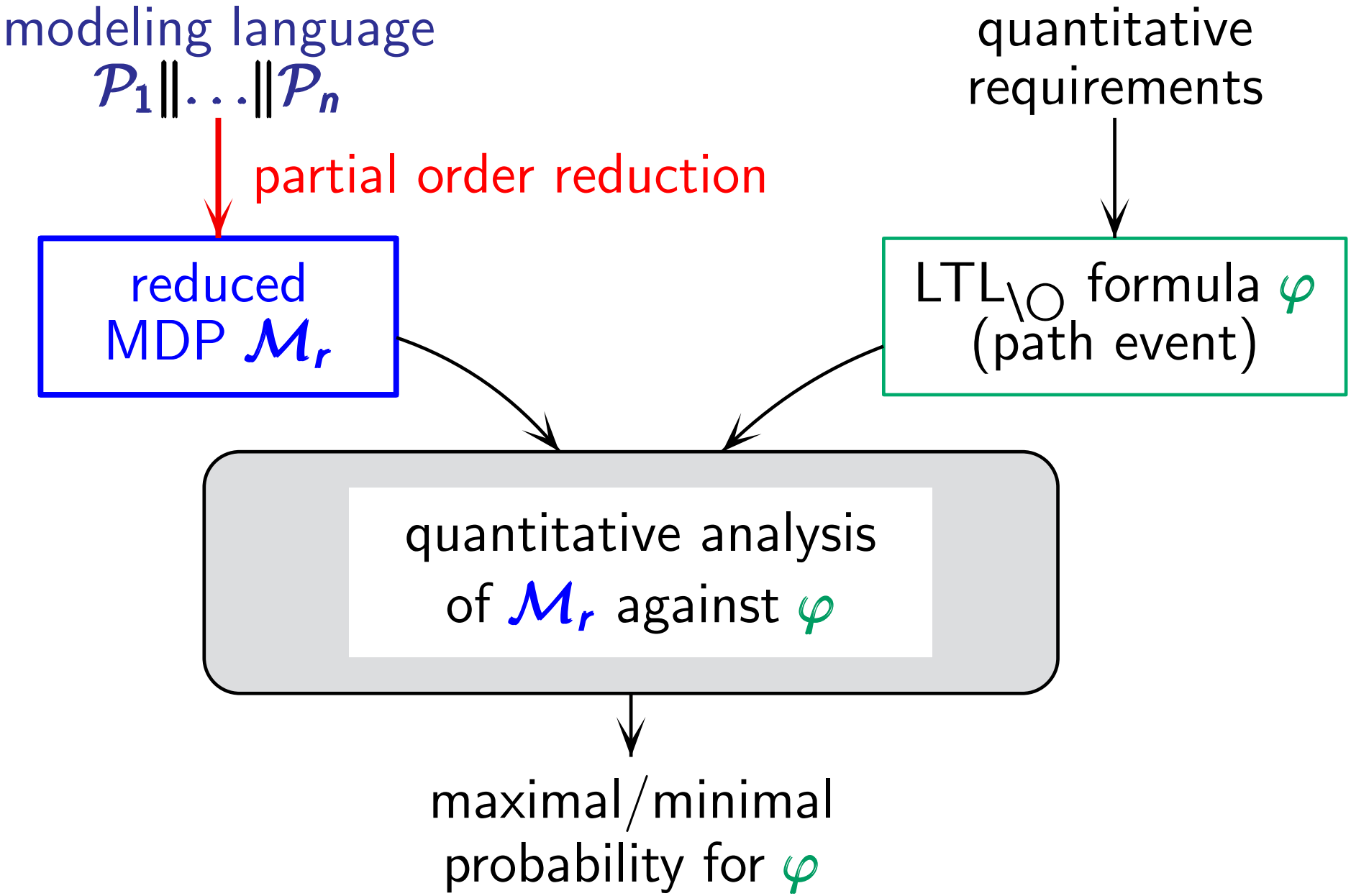
If (A1)-(A4) hold then $\mathcal{M}$ and $\mathcal{M}_r$ have the same extremal probabilities for all stutter-invariant properties.

Probabilistic model checking



Probabilistic model checking, e.g., LiQuor



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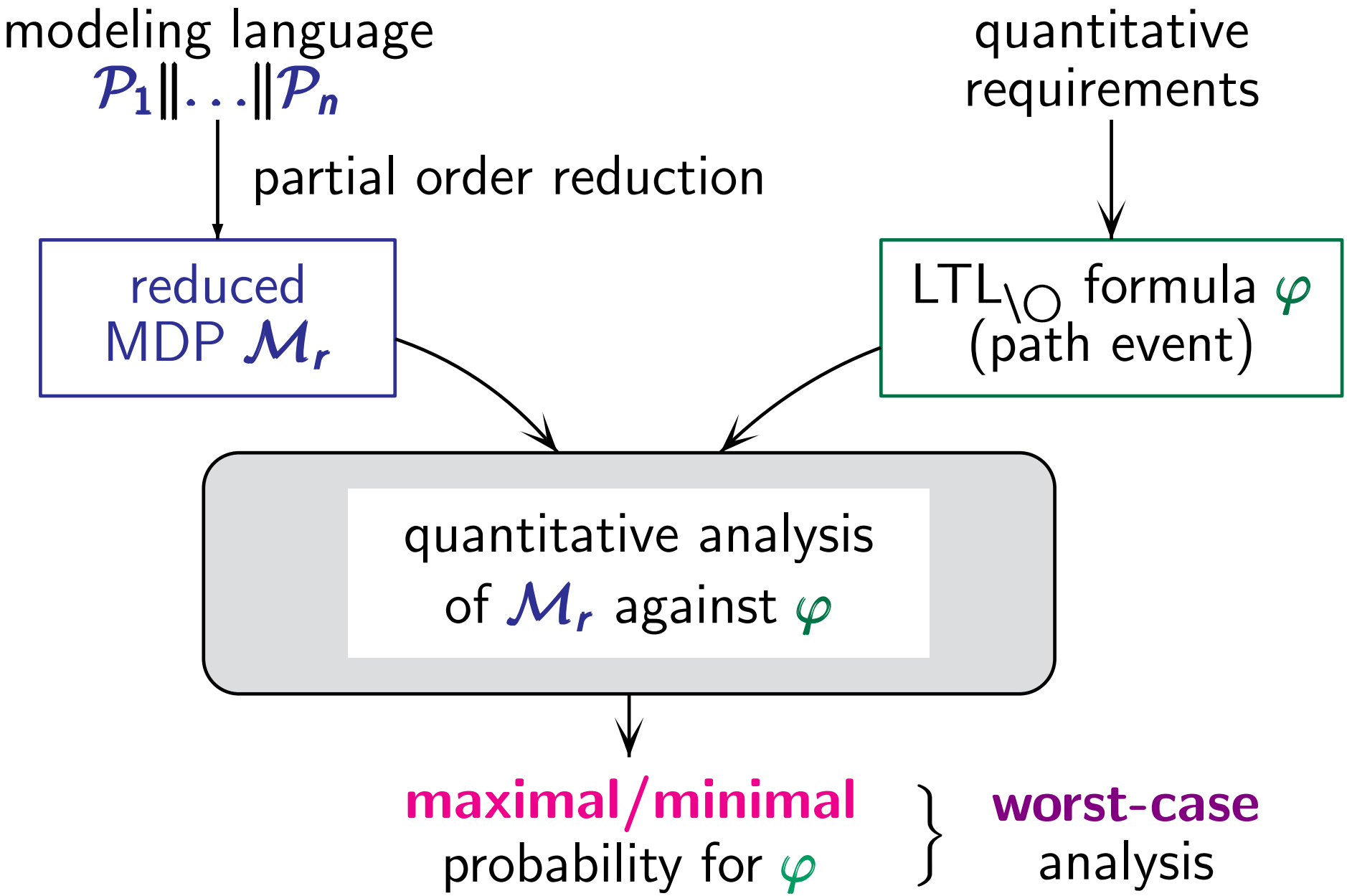
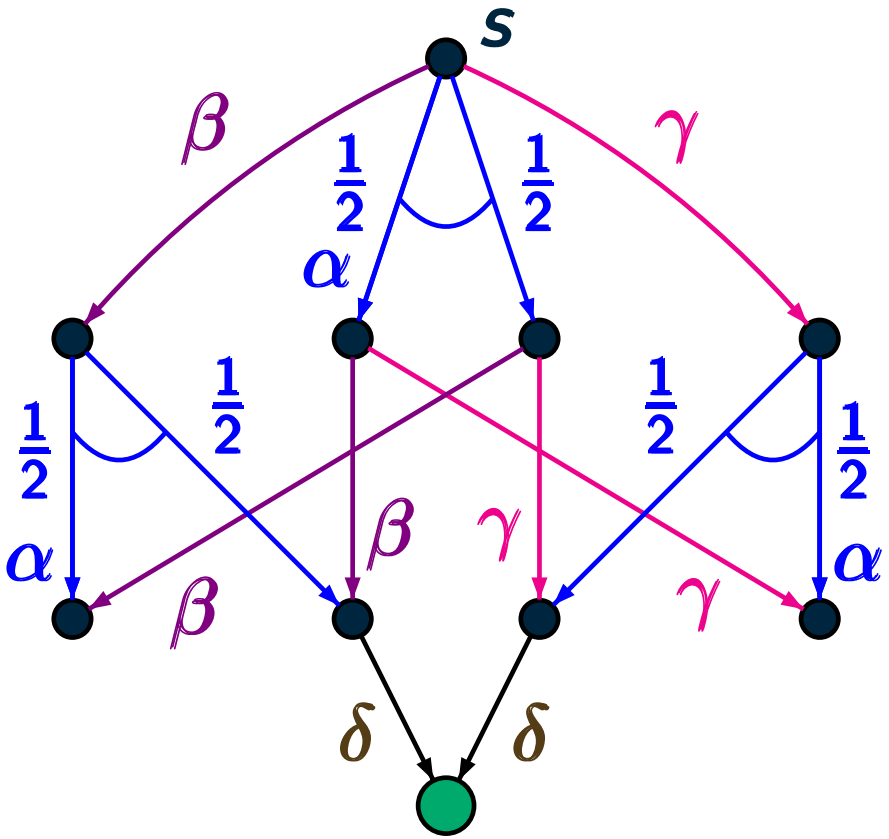


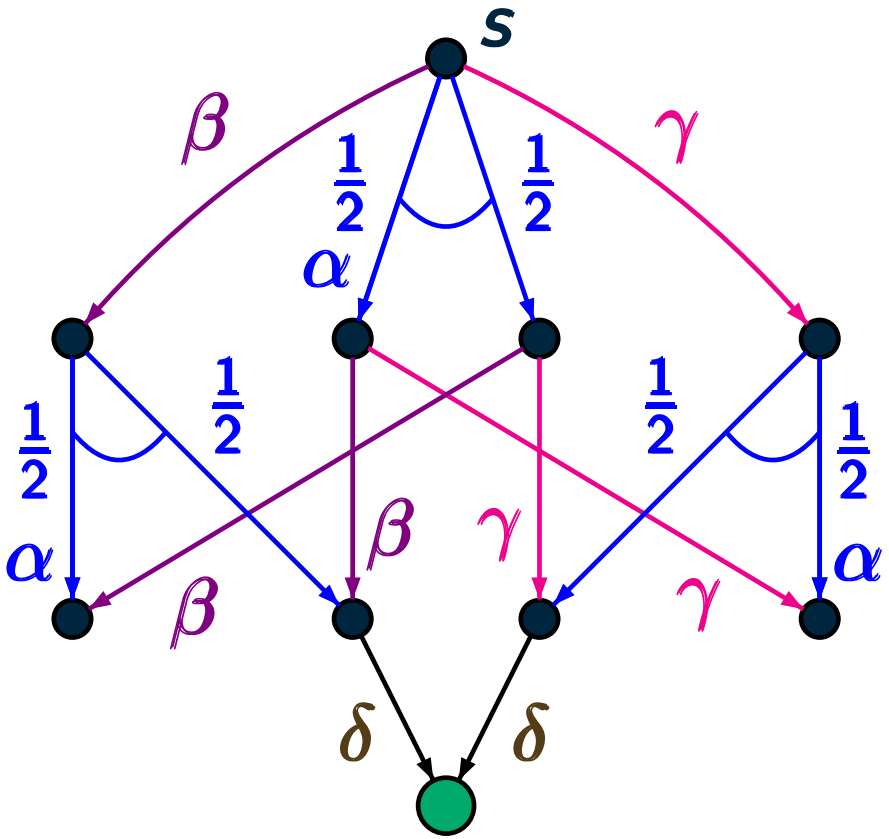
Example: extremal scheduler for MDP



original MDP $\mathcal{M}$

α independent
from β and γ

Example: extremal scheduler for MDP



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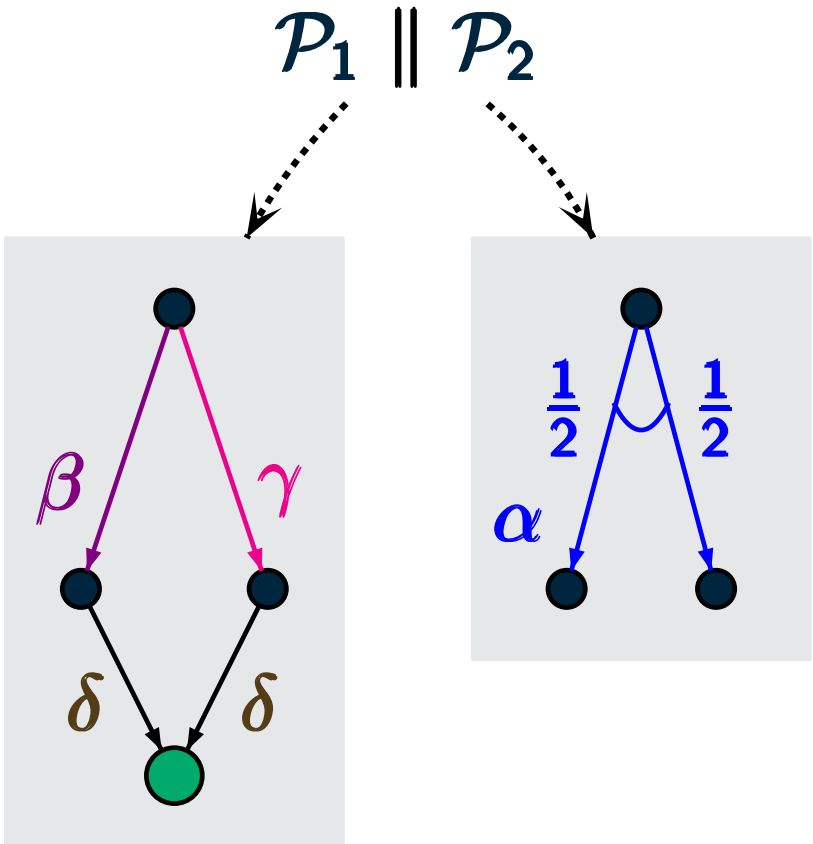
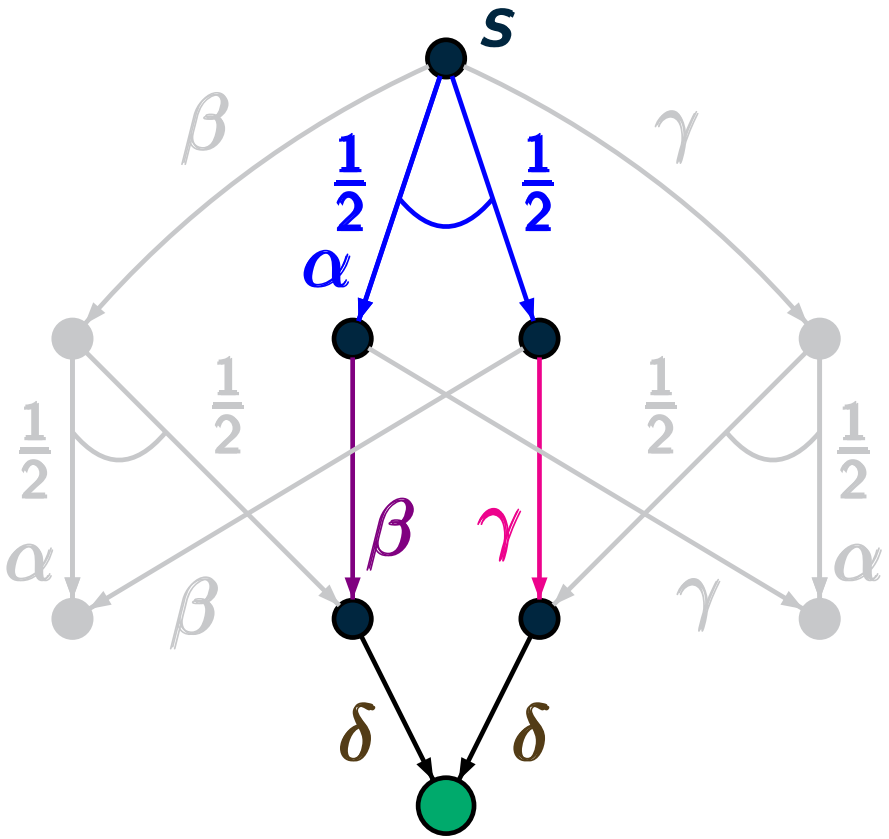


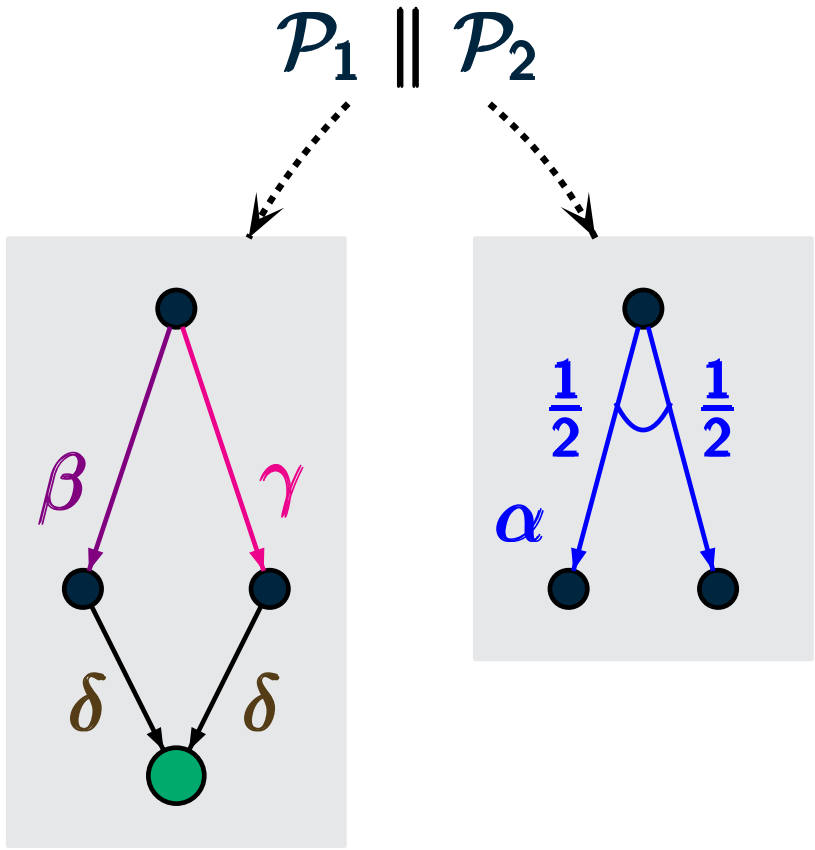
action α: $x := \mathit{random}(0, 1)$

action δ:

IF $x > 0$ THEN $y := 1$ FI

Example: extremal scheduler for MDP





scheduler chooses action β or γ of process $\mathcal{P}_1$, depending on $\mathcal{P}_2$'s internal probabilistic choice

Outline

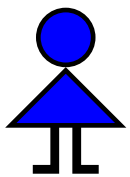
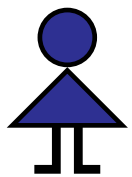
- Markov decision processes (MDP) and quantitative analysis against path events
- partial order reduction for MDP
- partially-oberservable MDP ←
- conclusions

Monty-Hall problem



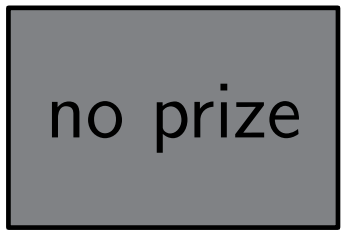
3 doors
initially closed

candidate

show
master

Monty-Hall problem



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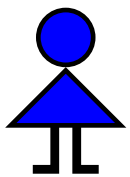
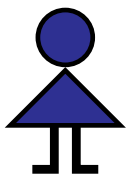
candidate





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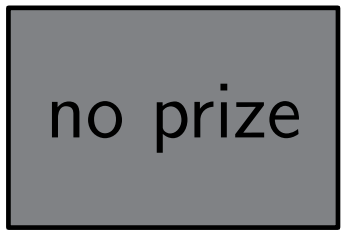

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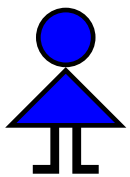
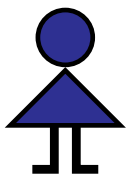
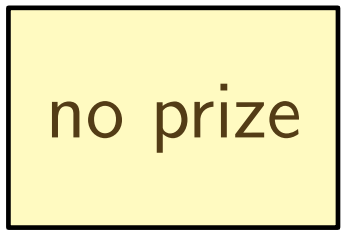




show
master

1. candidate chooses one of the doors

Monty-Hall problem



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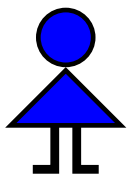
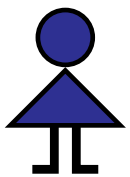
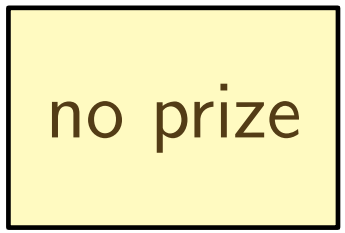




show
master

1. candidate chooses one of the doors
2. show master opens a non-chosen, non-winning door

Monty-Hall problem



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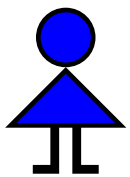

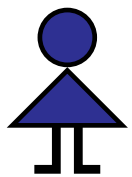

show
master

1. candidate chooses one of the doors
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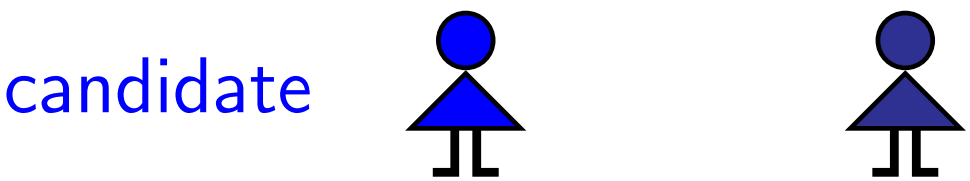
3. candidate has the choice:
   - keep the choice or
   - switch to the other (still closed) door

Monty-Hall problem



1. candidate chooses one of the doors
2. show master opens a non-chosen, non-winning door
3. candidate has the choice:
   - keep the choice or
   - switch to the other (still closed) door
4. show master opens all doors

Monty-Hall problem

3 doors
initially closed

optimal strategy for the candidate:

initial choice of the door: arbitrary
revision of the initial choice (switch)

probability for getting the prize: $\frac{2}{3}$

MDP for the Monty-Hall problem

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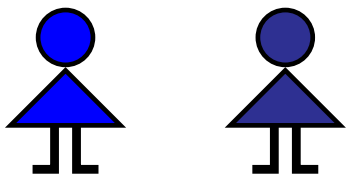
3 doors
initially closed

candidate's actions

1. choose one door
3. keep or switch **?**



show master's actions

2. opens a non-chosen, non-winning door
4. opens all doors

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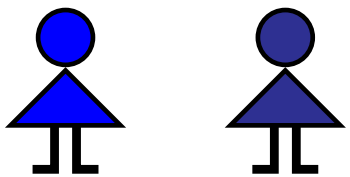

~~show master's actions~~

~~2. opens a non-chosen, non-winning door~~

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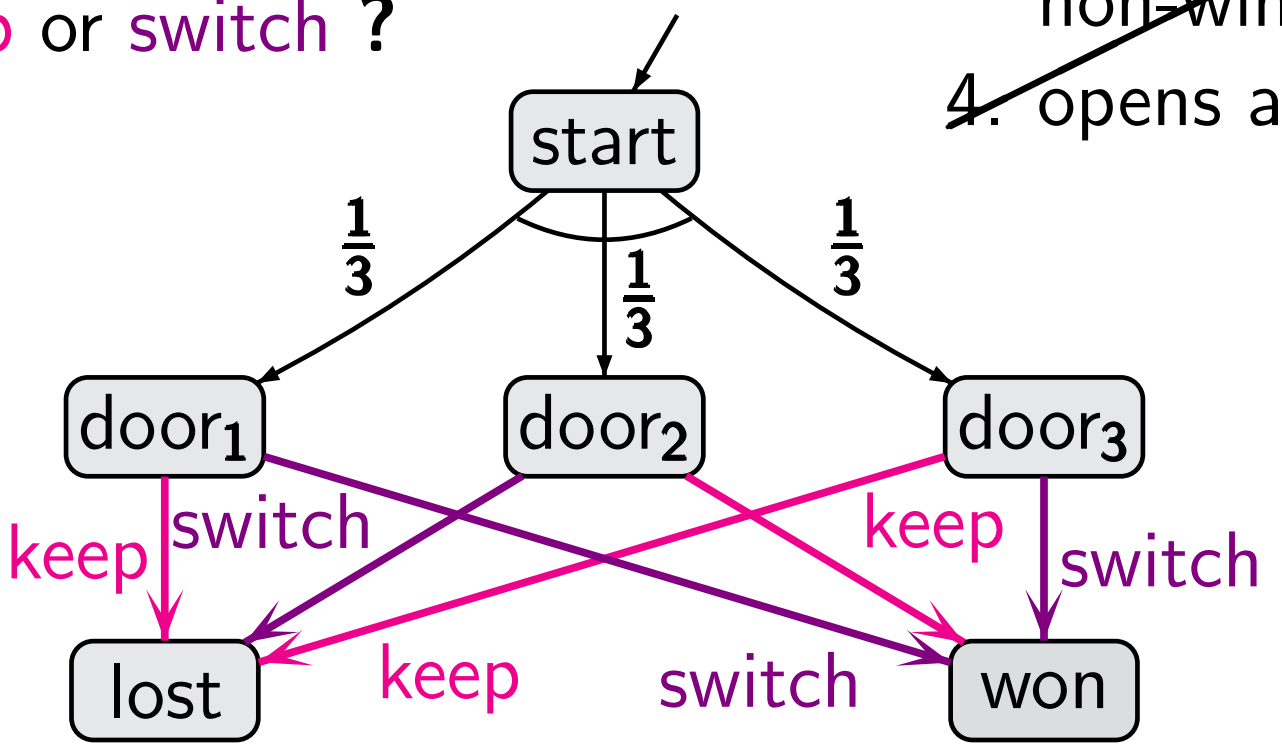

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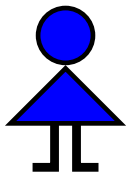
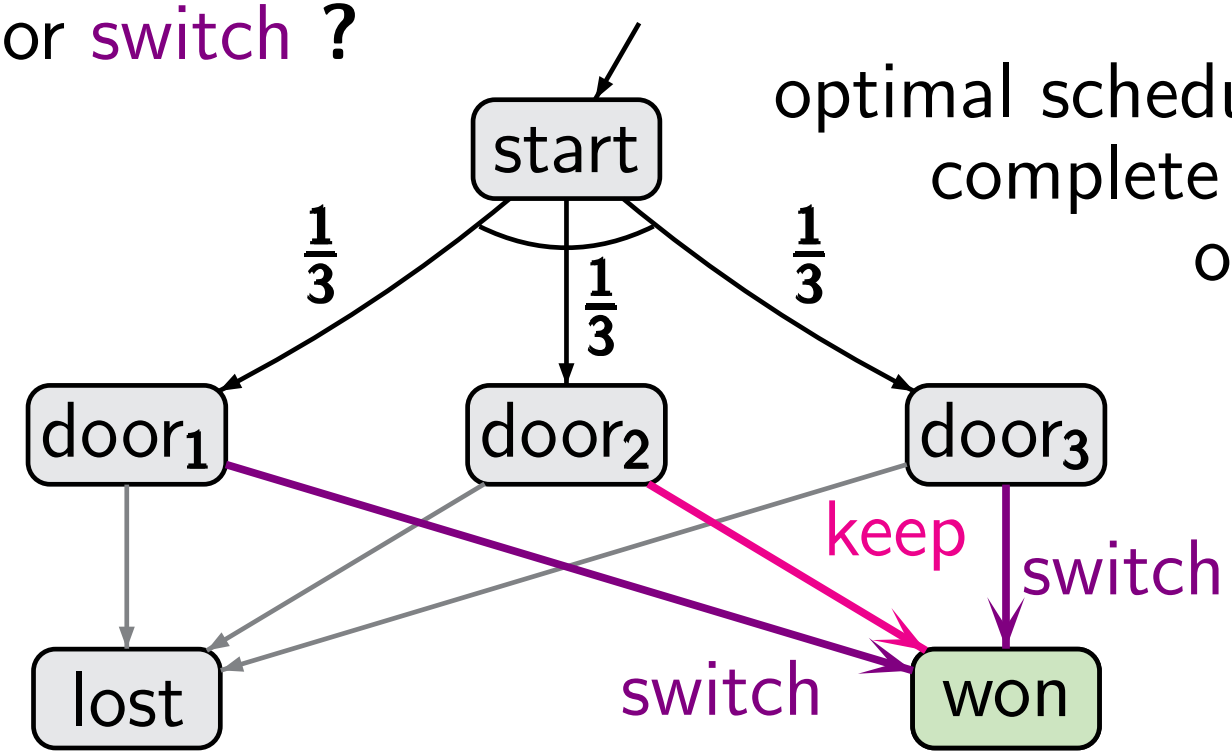


$\mathbf{Pr_{max}(start, \Diamond won) = 1}$

optimal scheduler requires complete information on the states



MDP for the Monty-Hall problem

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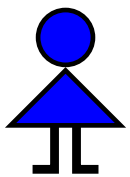
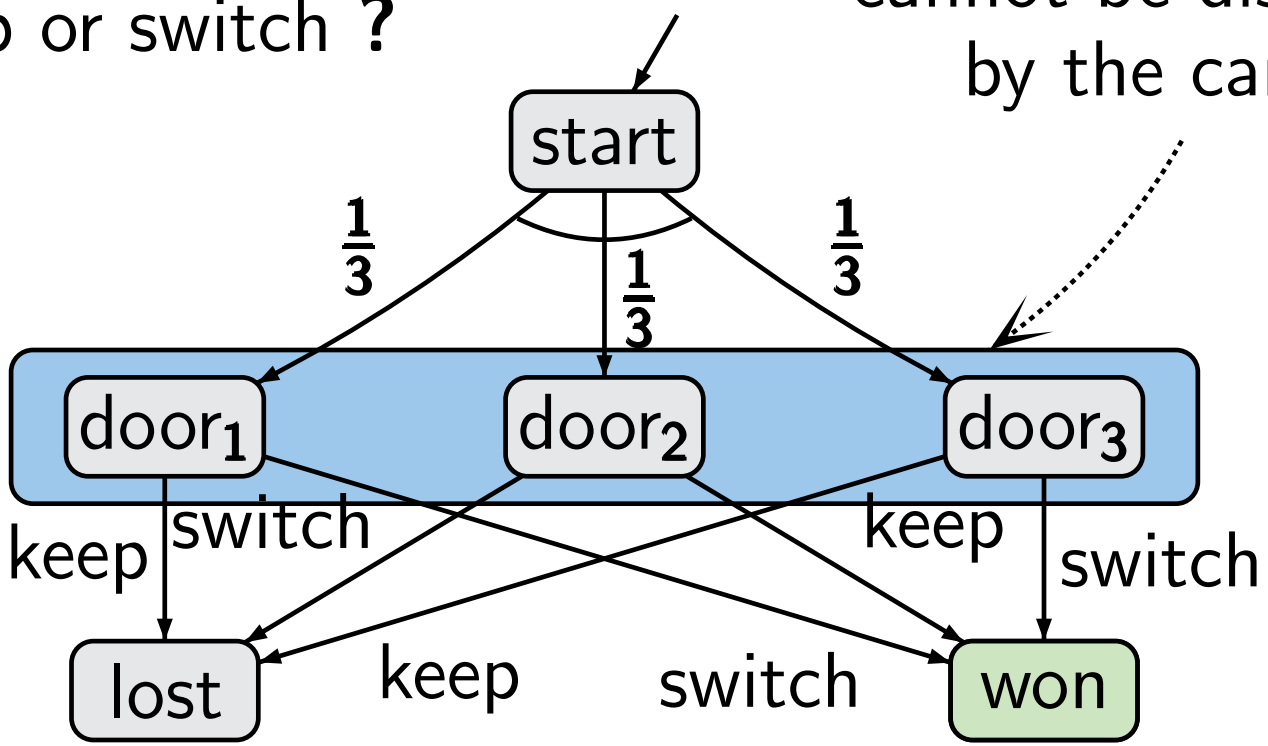


cannot be distinguished
by the candidate



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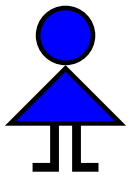
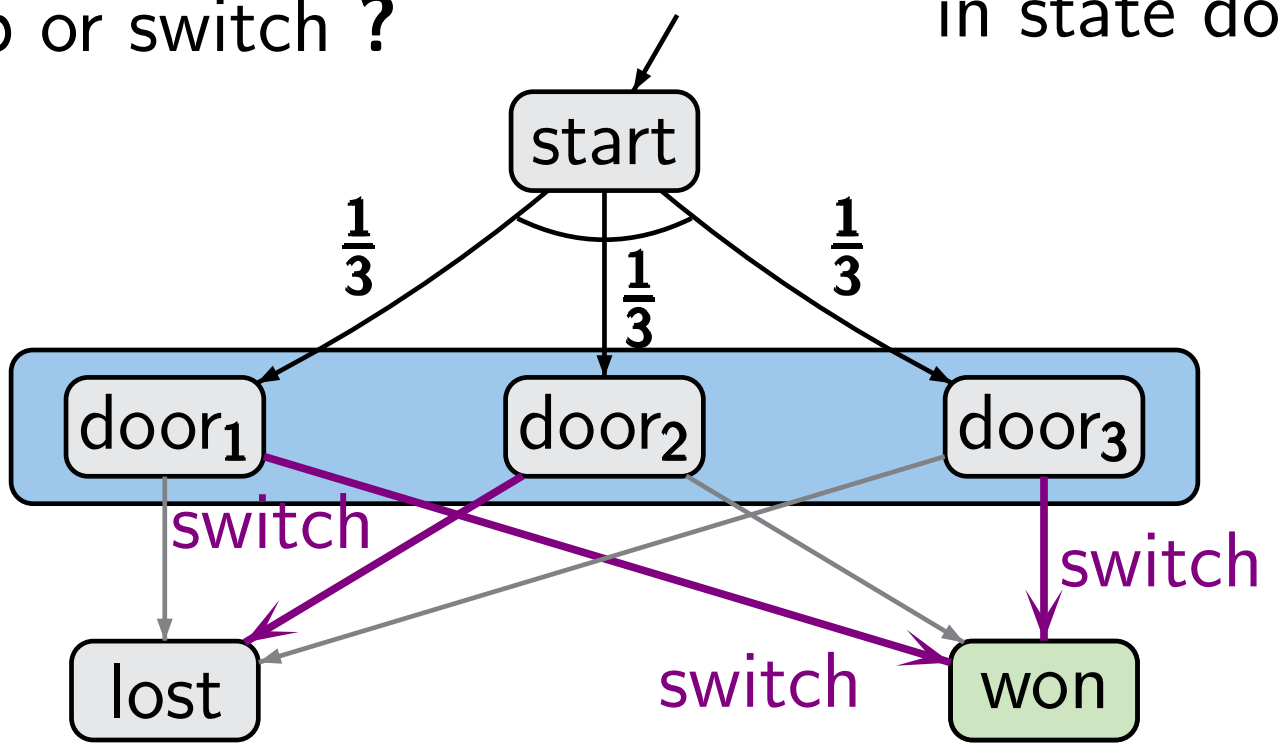


observation-based strategy:

choose action switch
in state door$_i$



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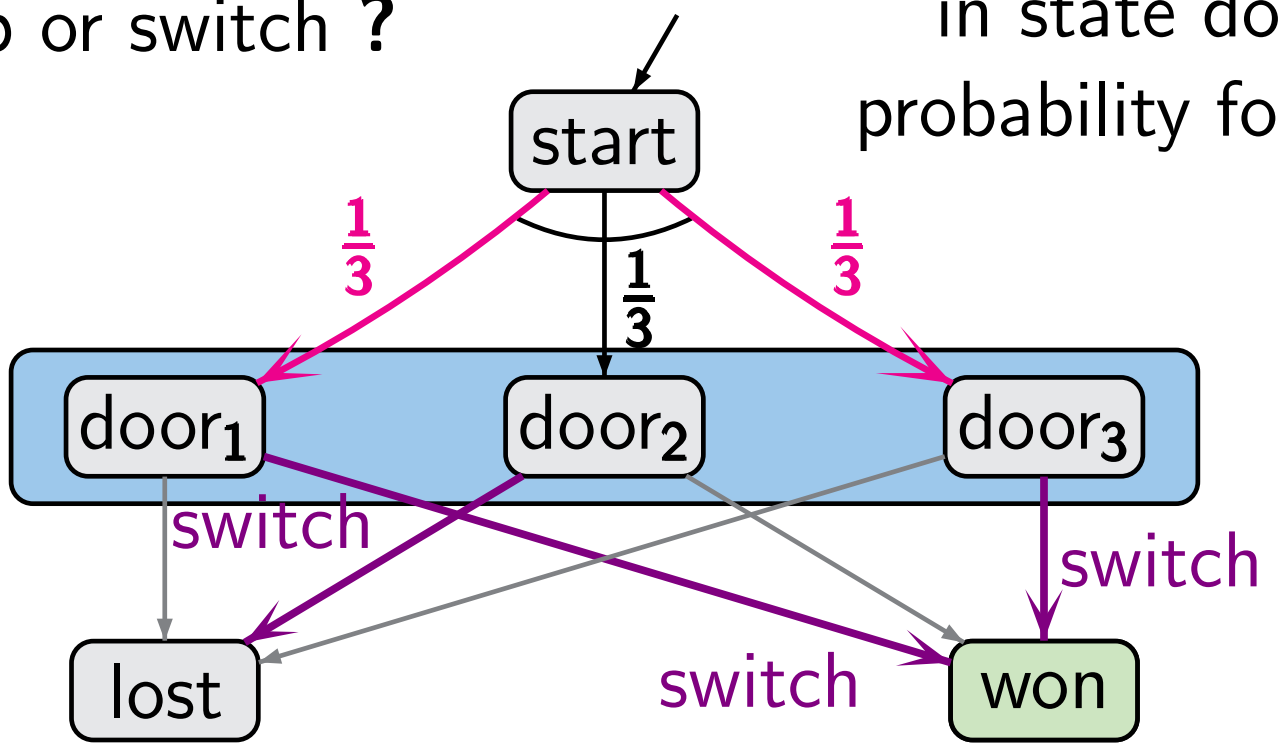
probability for $\Diamond$ ***won***: $\frac{2}{3}$



Partially-observable Markov decision process

A partially-observable MDP (POMDP for short) is an MDP $\mathcal{M} = (S, Act, P, \ldots)$ together with an equivalence relation $\sim$ on S

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if $s_1 \sim s_2$ then s_1, s_2 cannot be distinguished from outside (or by the scheduler)

observables: equivalence classes of states

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observables: equivalence classes of states

observation-based scheduler:

scheduler $D : S^+ \to Act$ such that for all $\pi_1, \pi_2 \in S^+$:

$$D(\pi_1) = D(\pi_2) \quad \text{if} \quad obs(\pi_1) = obs(\pi_2)$$

where $obs(s_0\, s_1 \ldots s_n) = [s_0]\,[s_1] \ldots [s_n]$

Extreme cases of POMDP

extreme cases of an POMDP:

- $s_1 \sim s_2$ iff $s_1 = s_2$
- $s_1 \sim s_2$ for all s_1, s_2

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Probabilistic automata are special POMDP

extreme cases of an POMDP:

- $s_1 \sim s_2$ iff $s_1 = s_2$ $\longleftarrow$ standard MDP
- $s_1 \sim s_2$ for all s_1, s_2 $\longleftarrow$ probabilistic automata

note that for totally non-observable POMDP:

observation-based scheduler $\widehat{=}$ function $D : \mathbb{N} \to Act$ $\widehat{=}$ infinite word over Act

Undecidability results for POMDP

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undecidability results for PFA carry over to POMDP

maximum probabilistic reachability problem "does $\Pr^{obs}_{max}(\Diamond F) > p$ hold ?" $\widehat{=}$ non-emptiness problem for **PFA**

Undecidability results for POMDP

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  [Paz'71], [Madani/Hanks/Condon'99], [Giro/d'Argenio'07]

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- The model checking problem for POMDP and several qualitative properties is undecidable, e.g.,

repeated reachability with positive probability

"does $\mathbf{Pr}^{obs}_{max}(\Box\Diamond F) > 0$ hold ?"

$\Box\Diamond \widehat{=}$ "infinitely often"

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Many interesting verification problems for distributed probabilistic multi-agent systems are undecidable.

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repeated reachability with positive probability

"does $\mathbf{Pr}^{obs}_{max}(\Box\Diamond F) > 0$ hold ?"

... already holds for $\underbrace{\text{totally non-observable POMDP}}_{\text{probabilistic Büchi automata}}$

Probabilistic Büchi automaton (PBA)

$\mathcal{P} = (Q, \Sigma, \delta, \mu, F)$

- Q finite state space
- Σ alphabet
- $\delta : Q \times \Sigma \times Q \to [0,1]$ s.t. for all $q \in Q$, $a \in \Sigma$:

$$\sum_{p \in Q} \delta(q, a, p) \in \{0, 1\}$$

- initial distribution μ
- set of final states $F \subseteq Q$

Probabilistic Büchi automaton (PBA)

$\mathcal{P} = (Q, \Sigma, \delta, \mu, F)$ ⟵ POMDP where $\Sigma = Act$ and $\sim \mathrel{\widehat{=}} Q \times Q$

- Q finite state space
- Σ alphabet
- $\delta : Q \times \Sigma \times Q \to [0,1]$ s.t. for all $q \in Q$, $a \in \Sigma$:

$$\sum_{p \in Q} \delta(q,a,p) \in \{0,1\}$$

- initial distribution μ
- set of final states $F \subseteq Q$

Probabilistic Büchi automaton (PBA)

$\mathcal{P} = (Q, \Sigma, \delta, \mu, F)$ ⟵ POMDP where $\Sigma = Act$ and $\sim \;\widehat{=}\; Q \times Q$

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$\mathbf{Pr}(x) =$ probability for the accepting runs for x

↑

accepting run: visits F infinitely often

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probability measure in the infinite Markov chain induced by x viewed as a scheduler

Accepted language of a PBA

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- Q finite state space, Σ alphabet
- $\delta : Q \times \Sigma \times Q \to [0, 1]$ s.t. ...
- initial distribution μ
- set of final states $F \subseteq Q$

three types of accepted language:

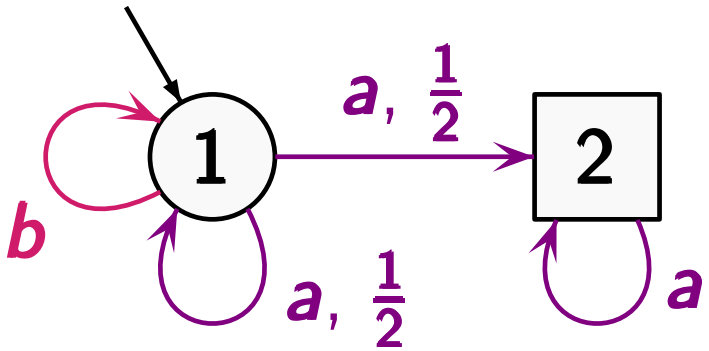
$$\mathcal{L}^{>0}(\mathcal{P}) = \{x \in \Sigma^\omega : \Pr(x) > 0\} \quad \text{probable semantics}$$

$$\mathcal{L}^{=1}(\mathcal{P}) = \{x \in \Sigma^\omega : \Pr(x) = 1\} \quad \text{almost-sure sem.}$$

$$\mathcal{L}^{>\lambda}(\mathcal{P}) = \{x \in \Sigma^\omega : \Pr(x) > \lambda\} \quad \text{threshold semantics}$$

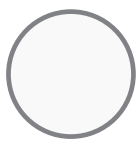
where $0 < \lambda < 1$

Example for PBA



initial state (probability 1)

final state



nonfinal state

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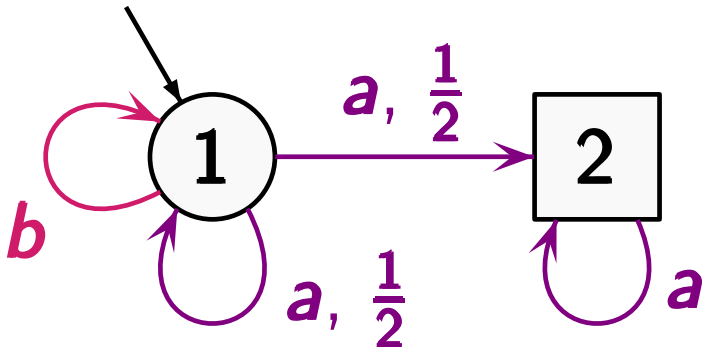

accepted language:

$$\mathcal{L}^{>0}(\mathcal{P}) \;=\; (a+b)^* a^\omega$$

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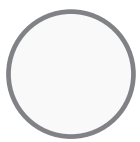

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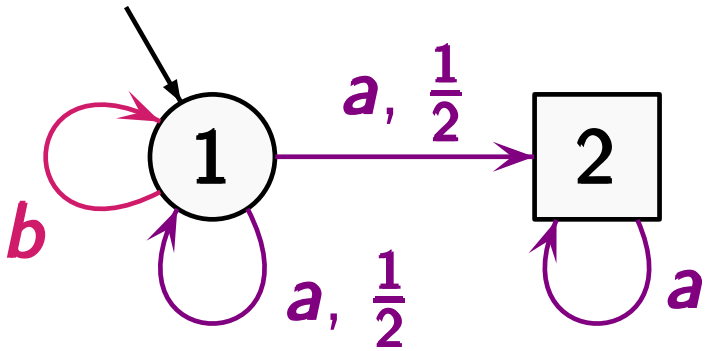

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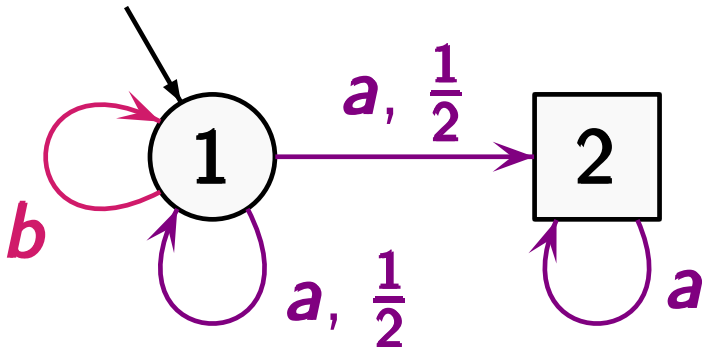

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Thus: **PBA$^{>0}$** are strictly more expressive than **DBA**

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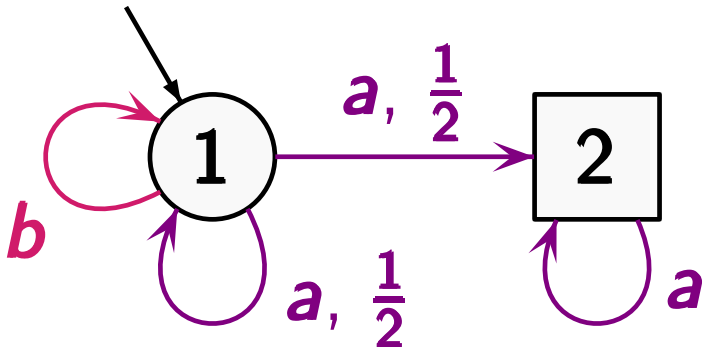

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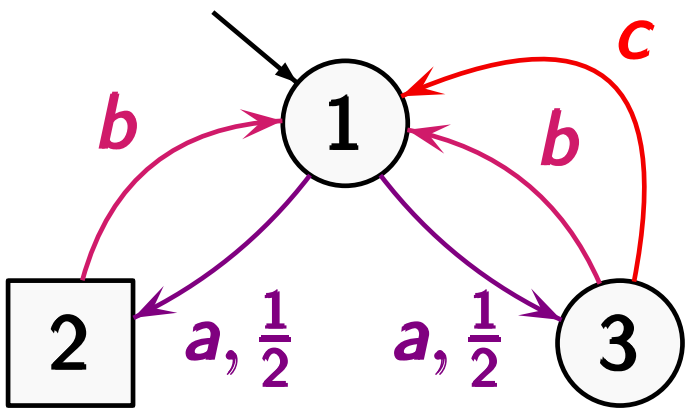

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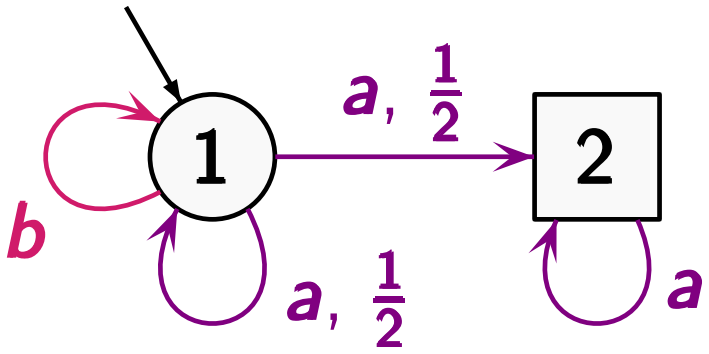

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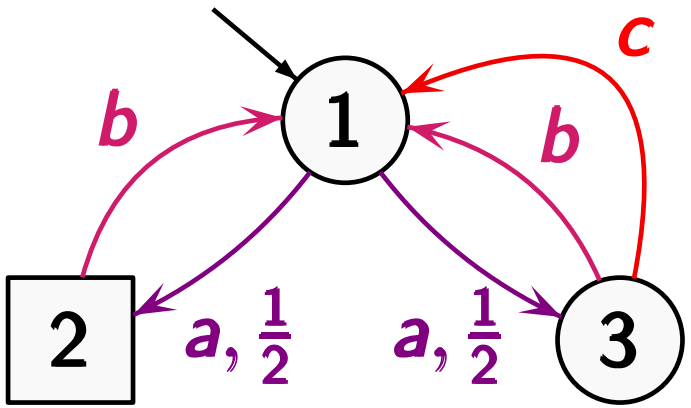

NBA accepts $((ac)^* ab)^\omega$

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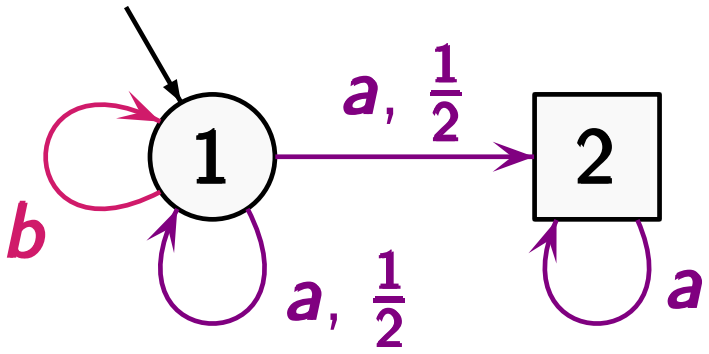

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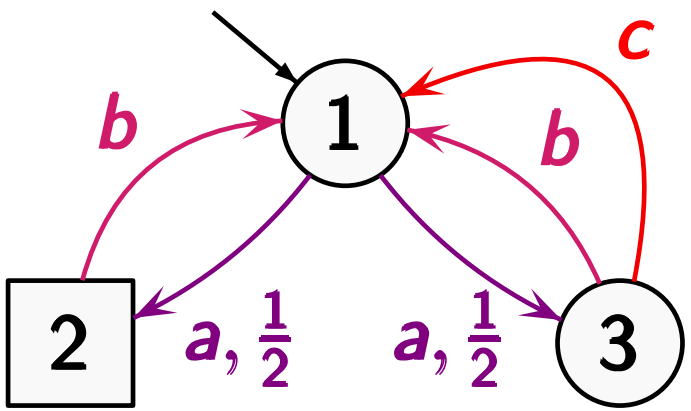

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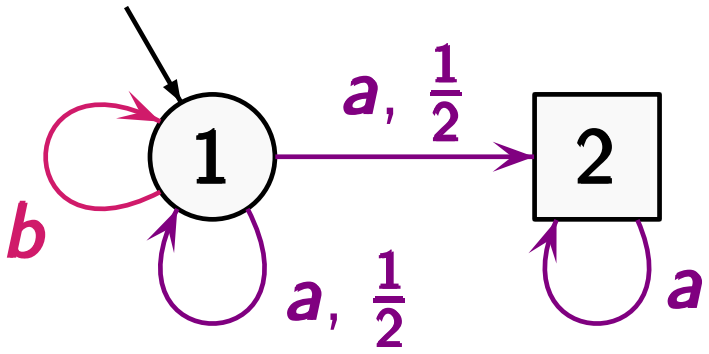

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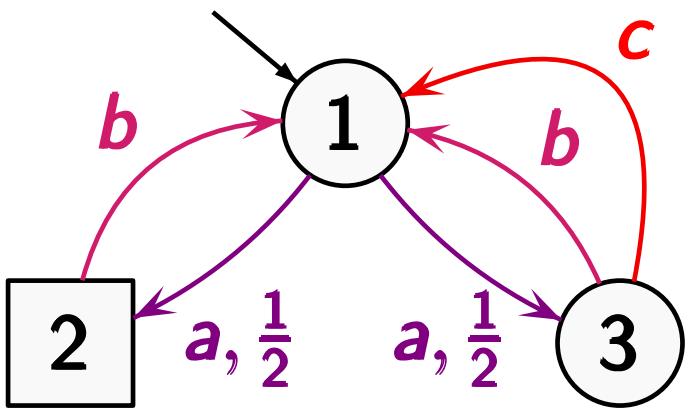

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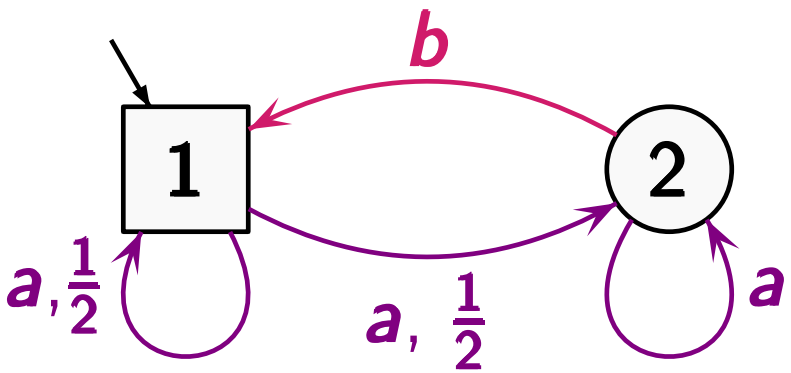

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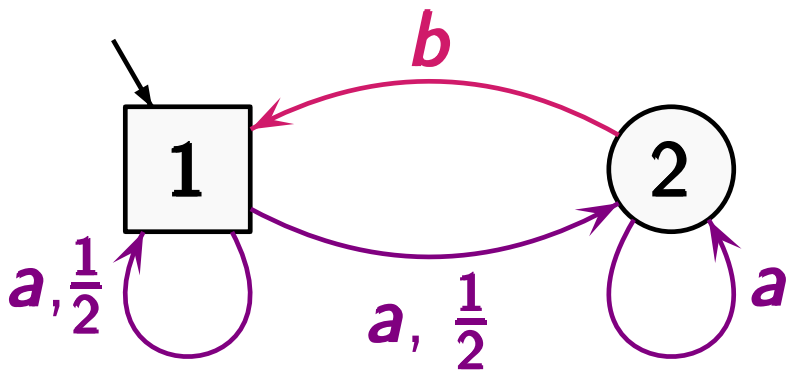

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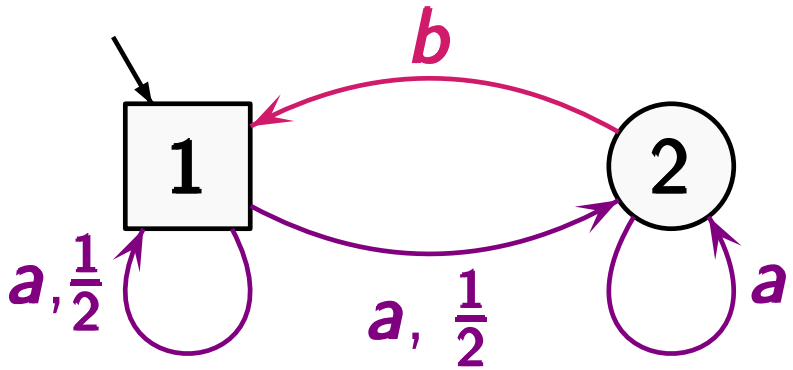

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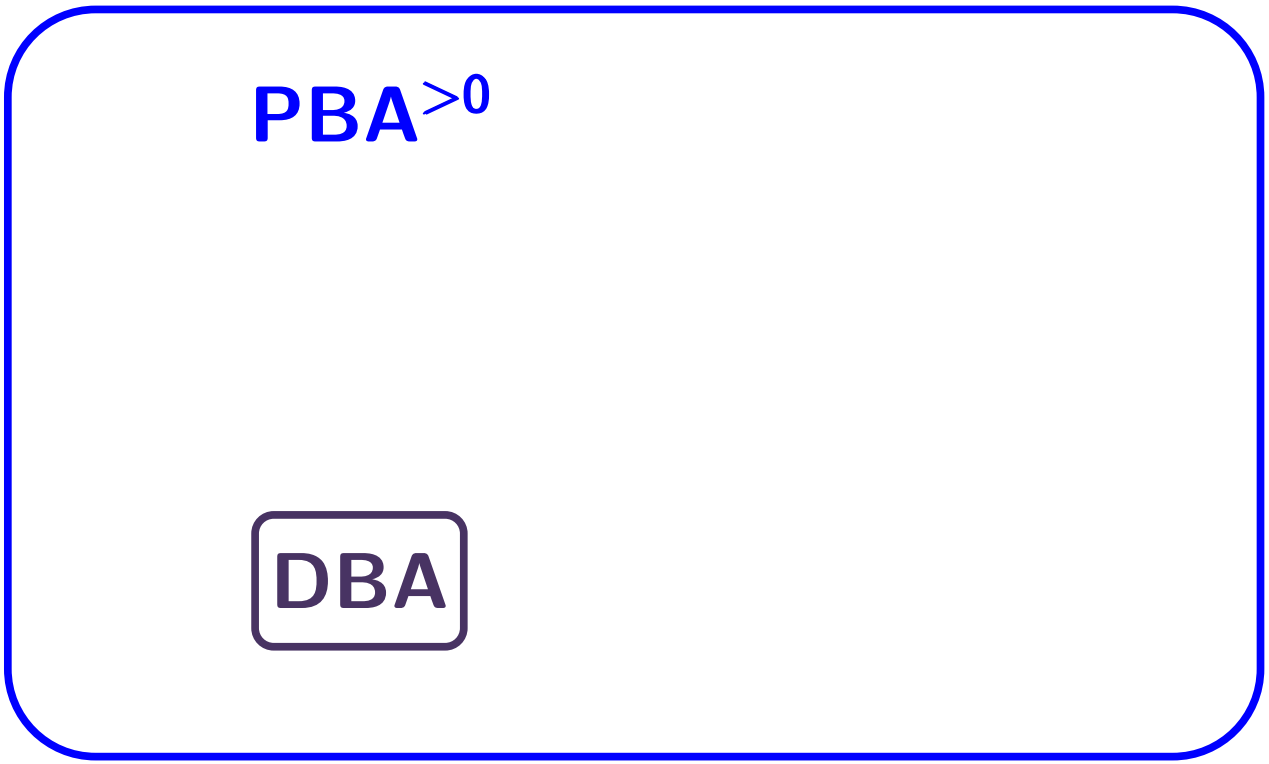
Expressiveness of PBA



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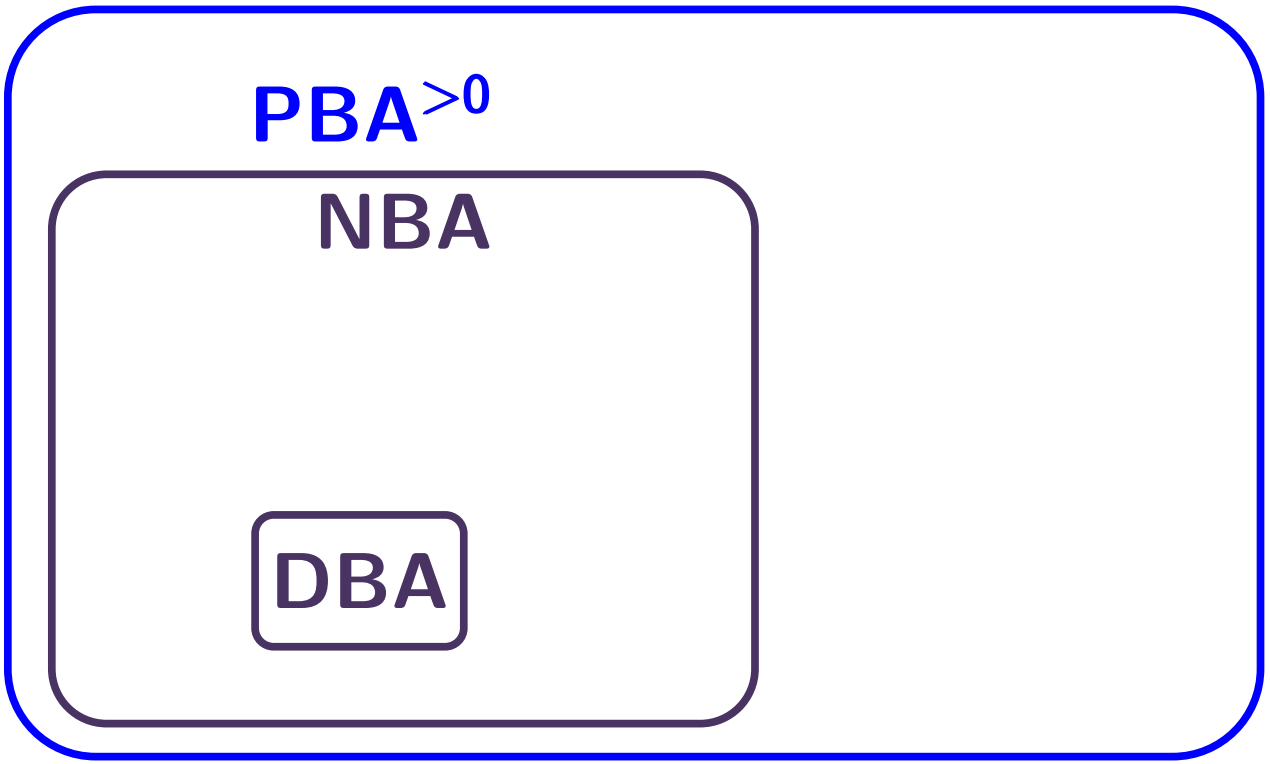

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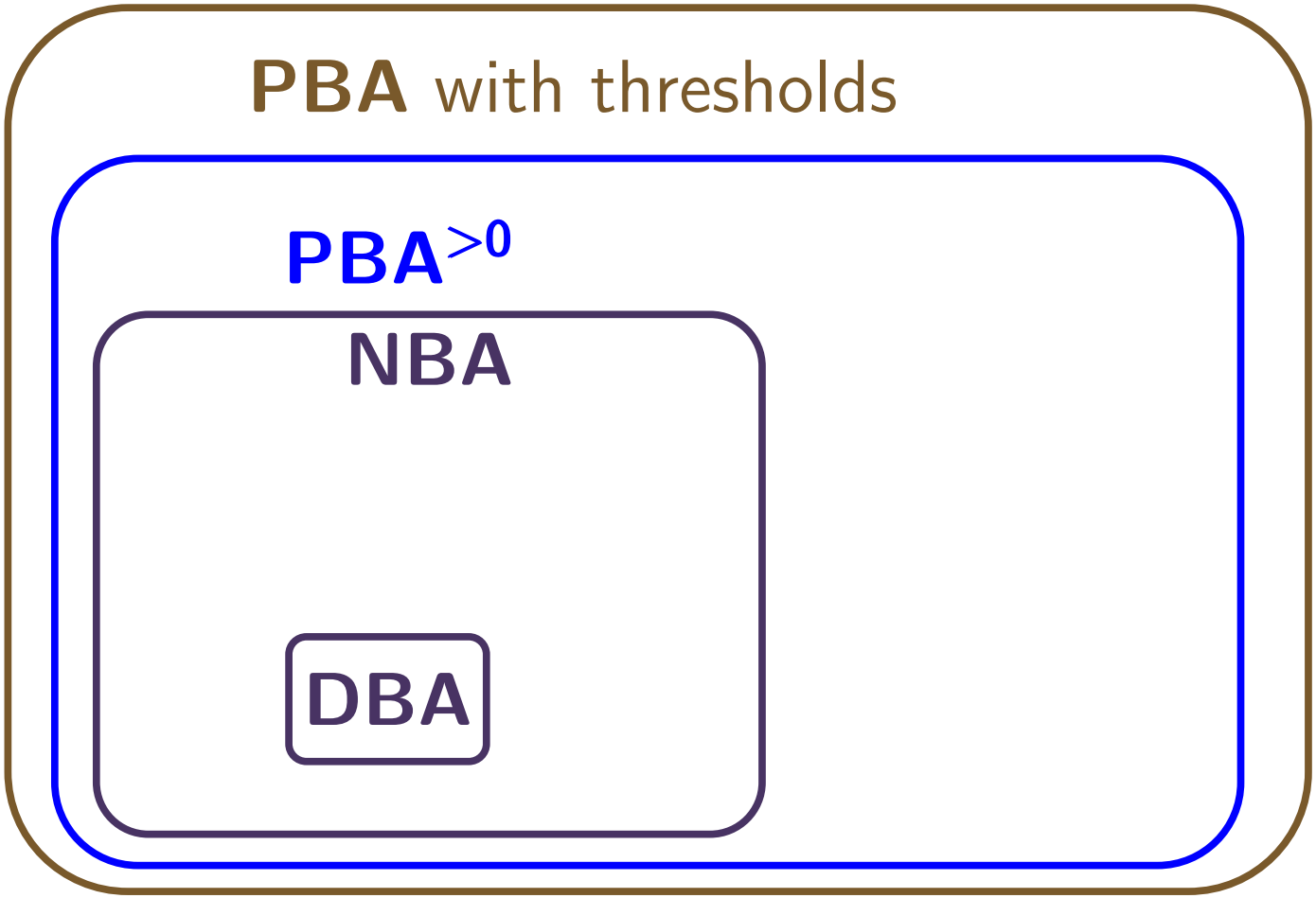

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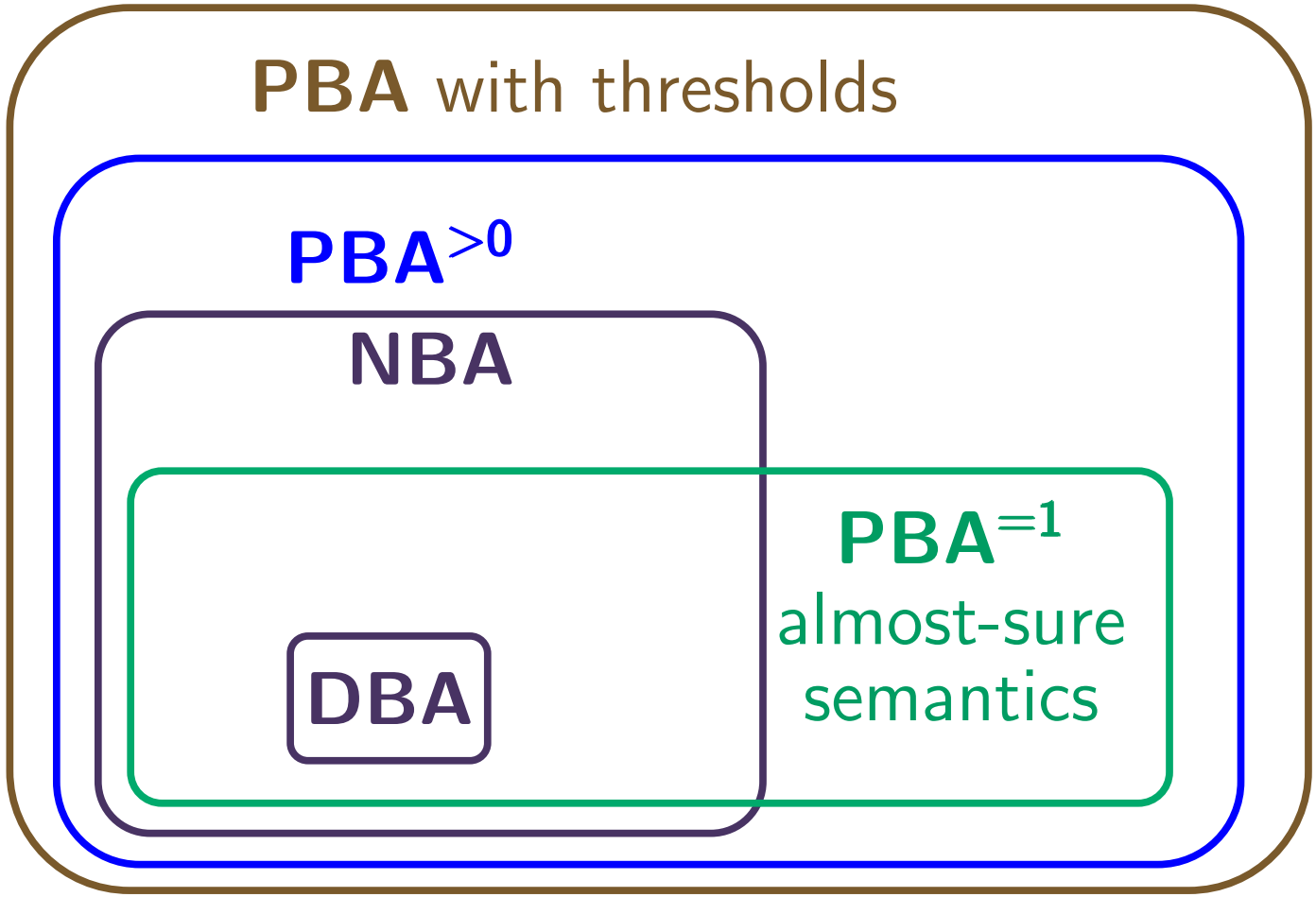

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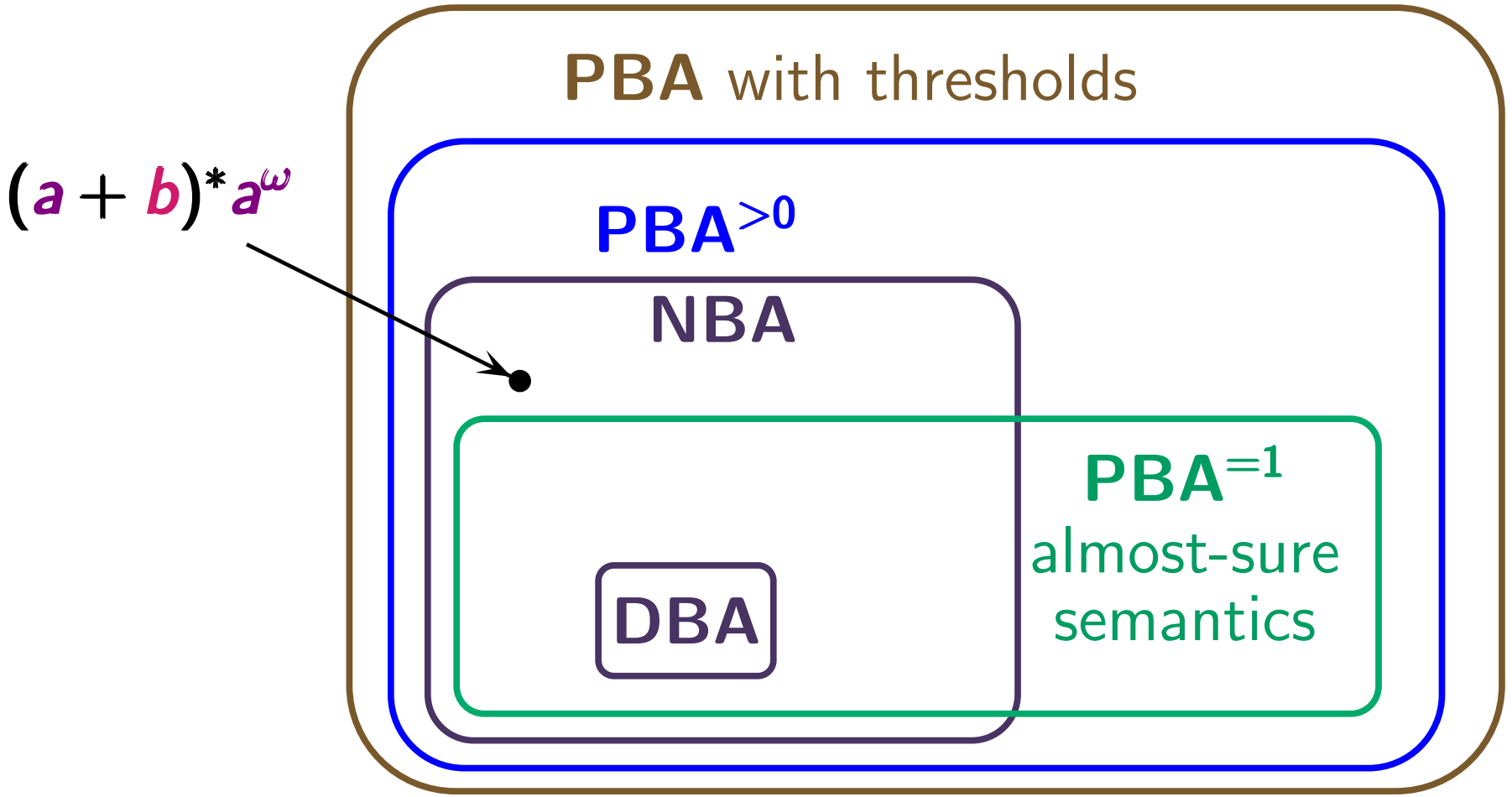

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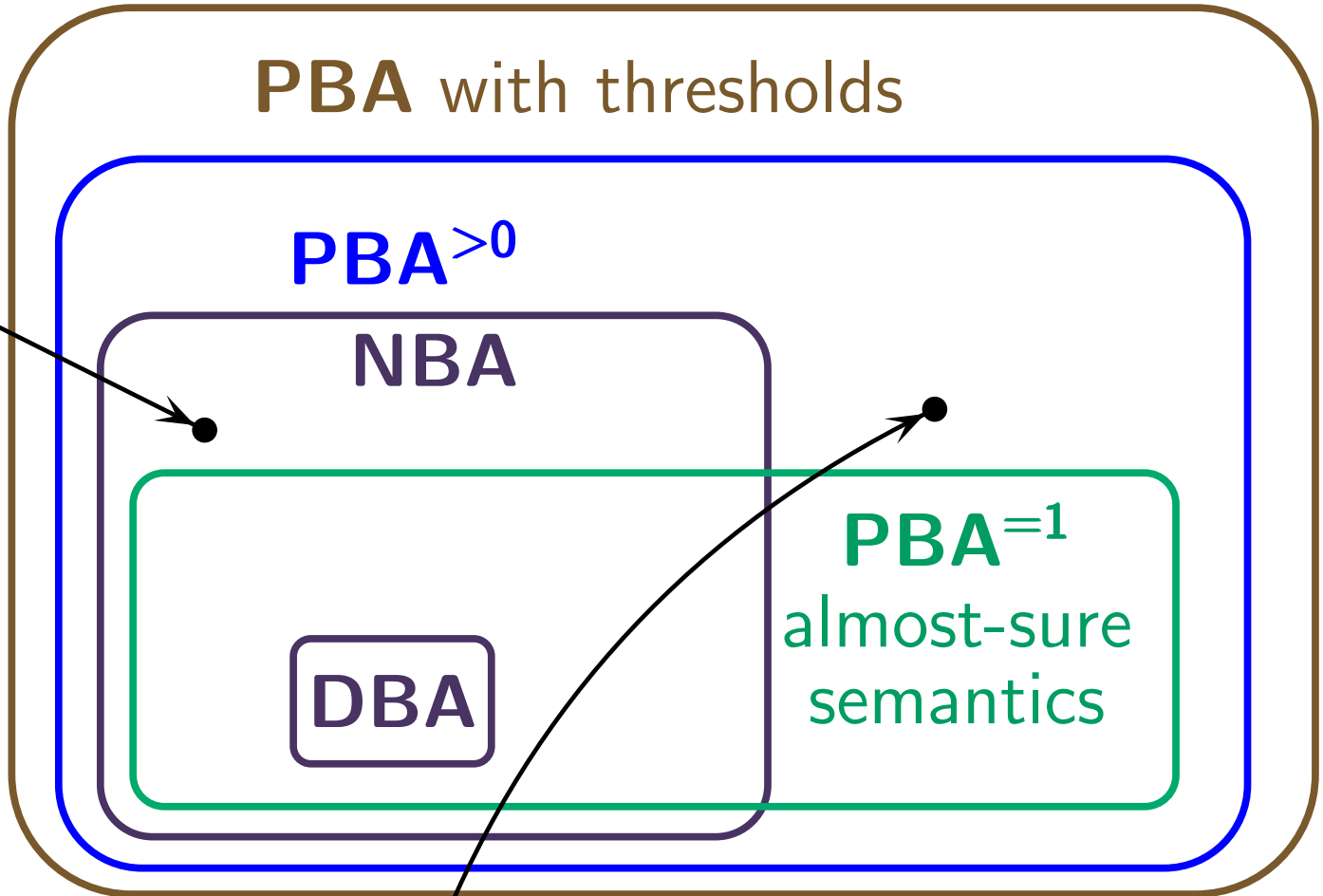

$(a+b)^*a^\omega$

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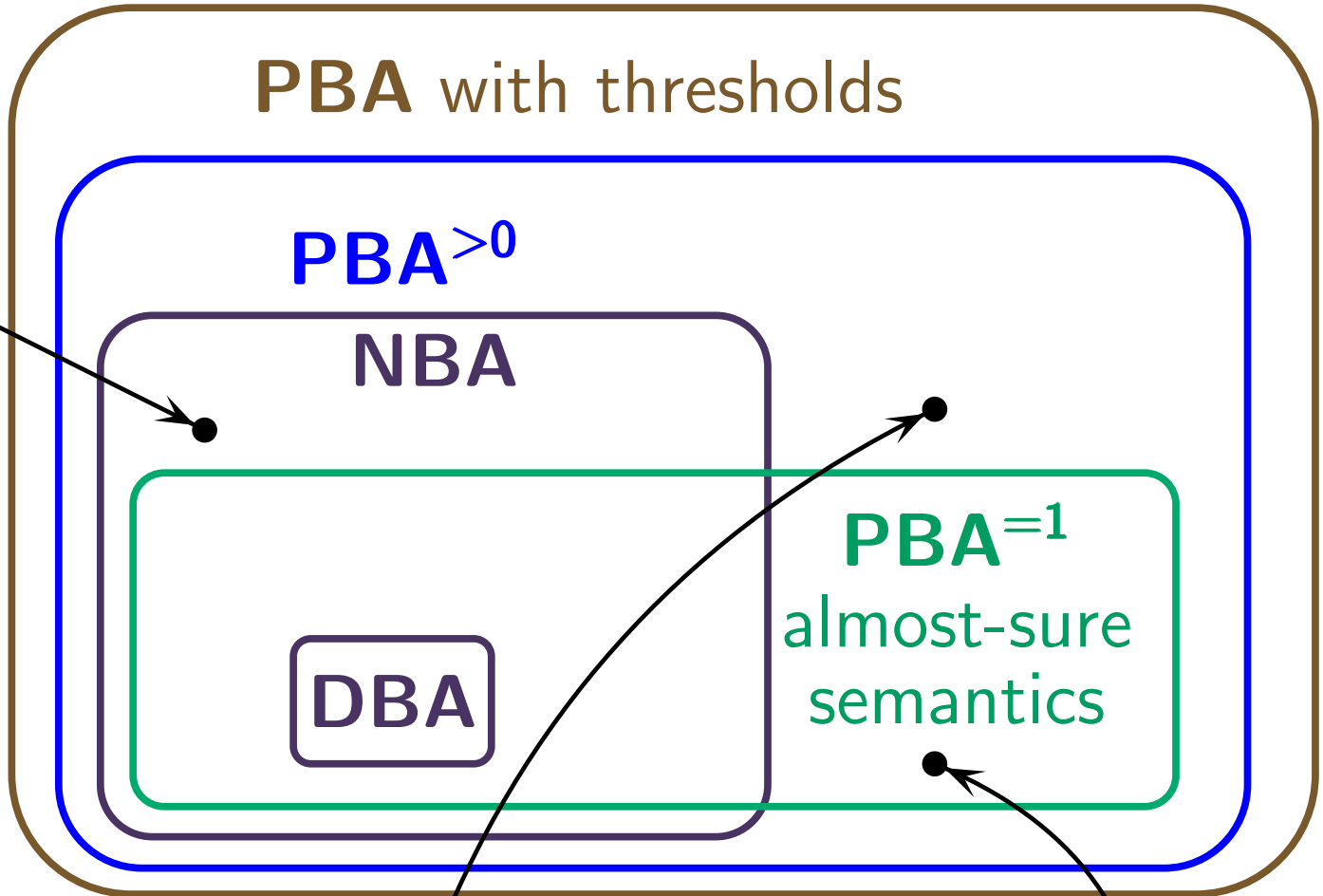

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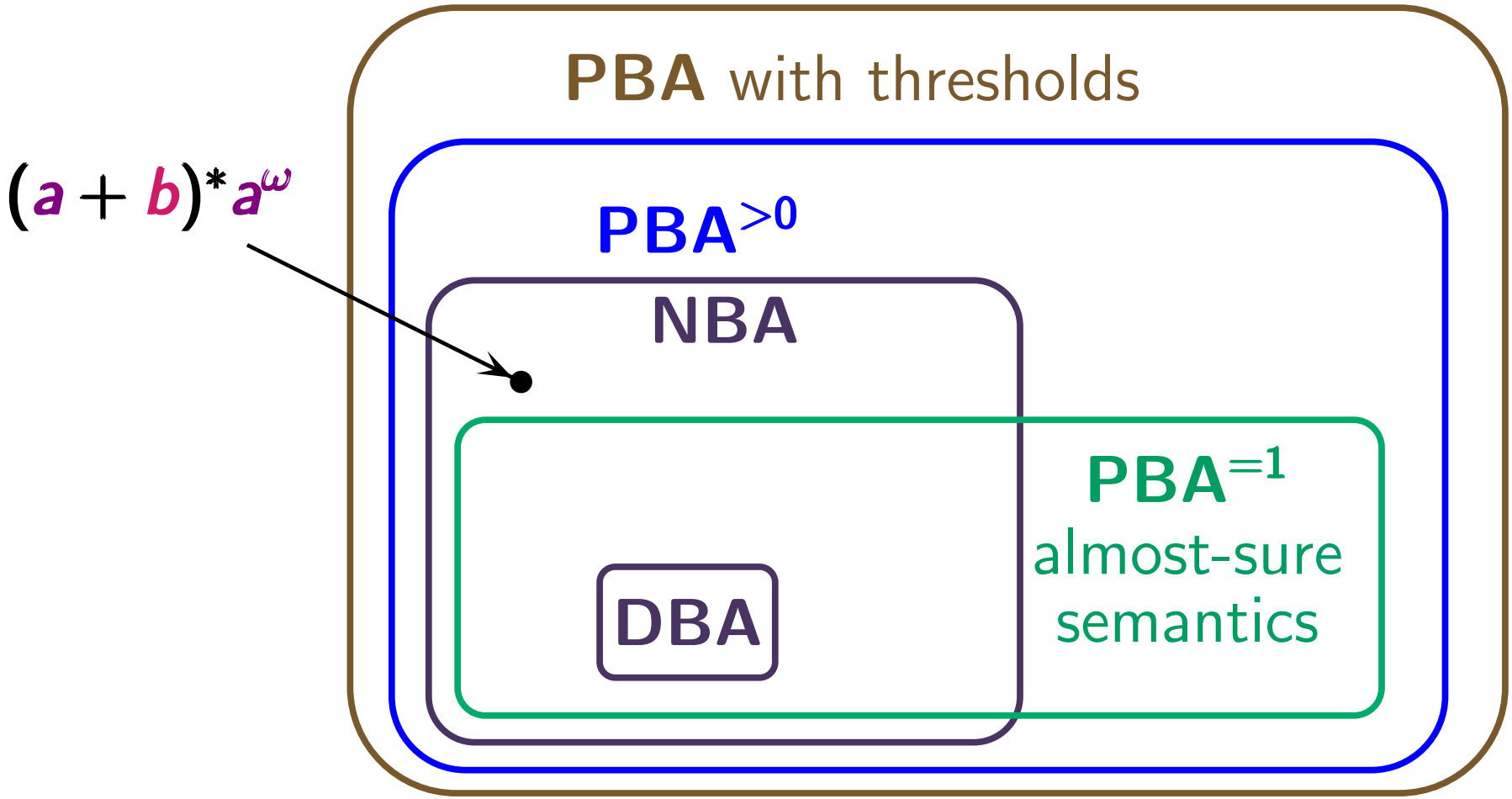

emptiness problem: undecidable for **PBA$^{>0}$**

decidable for **PBA$^{=1}$**

Outline

- Markov decision processes (MDP) and quantitative analysis against path events
- partial order reduction for MDP
- partially-oberservable MDP
- conclusions

Conclusion

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  - numerical methods for solving linear programs
  - known techniques for non-probabilistic systems

graph algorithms, LTL-2-AUT translators, ...
techniques to combat the state explosion problem
(such as partial order reduction)

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techniques to combat the state explosion problem
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but: strongly simplified definition of schedulers

assumption "full knowledge of the history" is inadequate, e.g., for agents of distributed systems

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proof via probabilistic language acceptors (PFA/PBA)

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