Reverse Mathematics & Nonstandard Analysis:

Making sense of infinite computations

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Abstract. We introduce a new notion of computability based on Nonstandard Analysis and prove it is equivalent to the usual definition. As an application, we obtain several deep results in Reverse Mathematics, a field closely related to Computability Theory. In particular, using Reverse Mathematics, we show that Nonstandard Analysis yields a 'more computable' framework for calculus than the Weierstraß ε-δ method. Finally, we demonstrate a concrete connection between Nonstandard Analysis and Bishop-style constructive analysis.

Outline

Reverse Mathematics (RM) is a program in the foundations of Mathematics initiated by Friedman ([1]) and developed extensively by Simpson ([2]). Its aim is to determine which *minimal* axioms prove theorems of ordinary Mathematics. Nonstandard Analysis (NSA) plays an important role in RM ([3,4]). We have previously considered RM where equality is replaced by the predicate $\approx$, i.e. equality up to infinitesimals from NSA ([5]). In particular, a 'copy' of the Reverse Mathematics for WKL_0 was obtained in ERNA, a weak system of NSA based on $I\Delta_0 + \exp$.

Using NSA, we introduce a new notion of computability and show that the new definition is equivalent to the usual one. Combining our results from RM and NSA, we observe that many theorems of ordinary Mathematics are computable when the nonstandard notions of continuity and differentiability are used in the theorem. However, when using the ε-δ notions of continuity and differentiability in these theorems, they become non-computable.

Acknowledgements

This research is made possible through the generous support of a grant from the John Templeton Foundation for the project *Philosophical Frontiers in Reverse Mathematics*. Please note that the opinions expressed in this publication are those of the author and do not necessarily reflect the views of the John Templeton Foundation

References

1. Harvey Friedman, *Systems of second order arithmetic with restricted induction, I & II (Abstracts)* , Journal of Symbolic Logic, 41, 1976, pp. 557–559
2. Stephen G. Simpson, *Subsystems of second order arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999
3. H. Jerome Keisler, *Nonstandard arithmetic and reverse mathematics*, Bull. Symbolic Logic, 12(1), 2006, pp. 100–125
4. Kazuyuki Tanaka, *The self-embedding theorem of* WKL_0 *and a non-standard method*, Ann. Pure Appl. Logic, 84(1), 1997, pp. 41–49
5. Sam Sanders, *ERNA and Friedman's Reverse Mathematics*, Journal of Symbolic Logic, 76, 2011