# Automorphisms of Computable Linear Orders and the Ershov Hierarchy.

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We consider the problem of determining, for computable linear orders of a given order type $\sigma$, the level in the arithmetical hierarchy at which rigidity breaks down. In particular we study the cases of $\sigma \in \{\mathbf{2}\cdot\eta, \omega+\zeta\}$. Kierstead proved in [Kie87] that, whereas every computable linear order of each such order type $\sigma$ has a $\Delta^0_2$ nontrivial automorphism, there exists a computable linear order of each of these order types which is $\Pi^0_1$ *rigid*—i.e. has no nontrivial $\Pi^0_1$ automorphism. We view these results in the light of the more specific question of where, in the arithmetical hierarchy extended via the Ershov hierarchy, rigidity breaks down. We approach this question by looking at a fundamental property displayed by the class of $\omega$-c.e. functions—which we label as $\mathcal{F}_\omega$. Define a class $\widehat{\mathcal{F}}$ of functions mapping $\omega$ into $\omega$ to be *exact uniform* $\Delta^0_2$ if $\widehat{\mathcal{F}}$ comprises precisely a list of functions $\{f_e\}_{e\in\omega}$ defined via a computable approximation $\{f_{e,s}\}_{e,s\in\omega}$—i.e. there exists a computable function $f$ such that $f(e,n,s) = f_{e,s}(n)$ and where, for all $e, n \in \omega$, $f_e(n)$ is defined to be $\lim_{s\to\infty} f_{e,s}(n)$ if $\lim_{s\to\infty} f_{e,s}(n)\downarrow$ and is undefined otherwise—which satisfies the property that, for all $e, n \in \omega$, if $\lim_{s\to\infty} f_{e,s}(n)\uparrow$ then $\liminf_{s\to\infty} f_{e,s}(n)$ tends to infinity. Now further define a class of functions $\mathcal{F}$ to be *uniform* $\Delta^0_2$ if $\mathcal{F} \subseteq \widehat{\mathcal{F}}$ for some exact uniform $\Delta^0_2$ class $\widehat{\mathcal{F}}$. We prove that given *any* exact uniform $\Delta^0_2$ class of functions $\widehat{\mathcal{F}}$ there exists a computable linear order of each order type $\sigma \in \{\mathbf{2}\cdot\eta, \omega+\zeta\}$ which is $\widehat{\mathcal{F}}$ rigid. We also show that $\mathcal{F}_\omega$ is uniform $\Delta^0_2$ and thus deduce that there exists a computable linear order of each of the above order types which is $\omega$-c.e. rigid. To conclude we will consider generalisations of these results. (Also we will explain our use of the term $\omega$-c.e. relative to a function $f$, as meaning that the *graph* of $f$ is $\omega$-c.e. in preference to the more restrictive notion—which we call *argument* $\omega$-c.e.—which applies to (total) $f$ if there exists a computable approximation $\{f_s\}_{s\in\omega}$ of $f$, and a computable function $h$, such that, for all $n \in \omega$, $\|\{\, s \mid f_{s+1}(n) \neq f_s(n) \,\}\| \leq h(n)$.)

## References

[Kie87] H.A. Kierstead. On $\Pi_1$-automorphisms of recursive linear orders. *Journal of Symbolic Logic*, 52(3):681–688, 1987.