The Partial Orderings of the Computably Enumerable *ibT*- and *cl*-Degrees are not Elementarily Equivalent

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A set A is *computably Lipschitz-(cl-)reducible* to a set B if it is Turing-reducible to B via a reduction where the oracle questions to determine $A(x)$ are bounded by $x + c$ for some constant c. If this constant can be chosen to be 0, then A is called *identity-bounded Turing-(ibT-)reducible* to B. Obviously, *ibT*-reducibility implies *cl*-reducibility and *cl*-reducibility implies weak-truth-table-(*wtt*-)reducibility, but the converse implications do not hold. For example, the 1-shift $A + 1 := \{x + 1 : x \in A\}$ of a noncomputable set A is always strictly *ibT*-reducible to A but *cl*-equivalent to A. Similarly, if f is a strictly increasing computable function that grows much faster than the identity, then the f-shift $\{f(x) : x \in A\}$ of a noncomputable set A is strictly *cl*-reducible to A but *wtt*-equivalent to A.

Considering the structures of the computably enumerable *wtt*-, *cl*- and *ibT*-degrees, a natural further question to ask is what the theories of these structures (in the first order language with the relation symbol $<$) look like and whether they are also different. Since the structure of the c.e. *wtt*-degrees is an upper semi-lattice with greatest element, no maximal pairs can exist in this structure, while in the c.e. *cl*- and *ibT*-degrees maximal pairs have been shown to exist [Barmpalias, 2005, and Fan and Lu, 2005]. Hence the theory of the c.e. *wtt*-degrees is clearly different from the theories of the c.e. *cl*- and *ibT*-degrees.

On the other hand, most known results about the c.e. *ibT*-degrees could be shown to hold for the c.e. *cl*-degrees as well. For example, the c.e. *ibT*-degrees are not dense [Barmpalias and Lewis, 2006], and, though the proof is more intricate, the same could later be shown for the c.e. *cl*-degrees [Day, 2010]. Another example are lattice embeddings. While we do not know whether all finite lattices are embeddable into the c.e. *ibT*- and *cl*-degrees, all finite lattices which are known to be embeddable into one of these degree structures are embeddable into both structures.

In this talk we indeed establish an elementary difference between the two theories under consideration by looking at cuppings. For two degrees $\mathbf{b} < \mathbf{a}$ (with respect to any fixed reducibility), we say that $\mathbf{b}$ *cups to* $\mathbf{a}$ if there is a degree $\mathbf{c} < \mathbf{a}$ such that $\mathbf{a}$ is the least upper bound of $\mathbf{b}$ and $\mathbf{c}$. We will show that for the *ibT*-degrees $deg(A + 1)$ is an upper bound for the degrees below $\mathbf{a}$ that do not cup to $\mathbf{a}$, while there is a *cl*-degree $\mathbf{a}$ such that the *cl*-degrees below it that do not cup to $\mathbf{a}$ do not have any upper bound less than $\mathbf{a}$.