Mass Problems and Relative Learnability

Kojiro Higuchi and Takayuki Kihara*

Mathematical institute, Tohoku University, Sendai, Miyagi, Japan
kihara.takayuki.logic@gmail.com

Cenzer-Kihara-Weber-Wu [1] found the notion of tree-immunity for closed subsets of Cantor space. Their method to prove some theorems on tree-immunity gives rise to a kind of "disjunction" under the limit-BHK interpretation of Limit Computable Mathematics (LCM) [2], a kind of constructive mathematics based on Learning Theory. Then the LCM-disjunctions are defined as binary operations on the subsets of Baire space. When disjunctive notions are represented as operations on the subsets of Baire space, this enable us to compare degrees of difficulty of disjunctive notions. This allows us to formalize the intuition that the intuitionistic disjunction is somehow more difficult than the classical one. We then introduce the notion of relative learnability for subsets of Baire space by defining the notion of learnable function. They are useful for understanding how does degrees of difficulty of disjunctive notions behave.

Based on this idea, we also introduce operators on the subsets of Baire space. For any mass problem $P \subseteq \mathbb{N}^{\mathbb{N}}$, new mass problems $P^{\triangledown}$, $P^{\triangledown_2}$, $P^{\triangledown_3}$, and $P^{\triangledown_4}$ are introduced, where they represent mass problems saying "solve P by a learning process with mind-change-bound 2", "solve P by a learning process", "solve P by a team-learning process by a team of two learners", and "solve P by a classic logical process", respectively. Note that $P \equiv_w P^{\triangledown} \equiv_w P^{\triangledown_2} \equiv_w P^{\triangledown_3} \equiv_w P^{\triangledown_4}$, where $\equiv_w$ denotes the Muchnik equivalence. For any nonempty Π^0_1 classes $P, Q \subseteq 2^{\mathbb{N}}$, we show the following: (1) If a computable function $\Gamma : P^{\triangledown} \to P$ exists, then P has a computable element. (2) If there is a finite covering $\{Q_i\}_{i<b}$ of $Q^{\triangledown_2}$ such that a computable funcion $\Gamma_i : Q_i \to P$ exists for any $i < b$, then P has a computable element. (3) If a learnable function $\Gamma : Q^{\triangledown_3} \to P$ exists, then P has a computable element. (4) If a team-learnable function $\Gamma : Q^{\triangledown_4} \to P$ exists, then P has a computable element. Note that $P^{\triangledown}$ is Π^0_1 whenever P is Π^0_1. Though $P^{\triangledown_3}$ is not Π^0_1 even if P is Π^0_1, there is a Π^0_1 class $P^{\blacktriangledown} \subseteq P^{\triangledown_3}$ saying "solve P by a learning process along T_P". For any Π^0_1 classes $P, R \subseteq 2^{\mathbb{N}}$ without computable element, we show the following: (5) If a computable function $\Gamma : P^{\triangledown} \otimes R \to P$ exists, then a computable function $\Delta : R \to P$ exists. (6) If a learnable function $\Gamma : P^{\blacktriangledown} \otimes R \to P$ exists, then a leanable function $\Delta : R \to P$ exists.

References

1. D. Cenzer, T. Kihara, R. Weber, and G. Wu, Immunity and non-cupping for closed sets, *Tbilisi Math. J.*, **2** (2009), pp. 77–94.
2. S. Hayashi, Mathematics based on incremental learning –excluded middle and inductive inference, *Theoret. Comput. Sci.* **350** (2006), no. 1, 125–139.

* Corresponding Author. This work was supported by Grant-in-Aid for JSPS fellows.