A remark on the existence of contact relations on Boolean algebras

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The notion of a contact algebra arises in mereotopology, which aims to describe properties of a topological space X by using as basic objects not the points of X, but certain subsets of X – called *regions* – and a binary relation C among these, called a *contact relation*, see e.g. [1] or [2,3] for details. The standard example is the Boolean algebra of regular closed sets of a Euclidean space with two regions being in contact if their intersection is not empty. More generally, a *Boolean contact algebra* is a structure $\langle B, +, \cdot, ^*, 0, 1, C\rangle$, where $\langle B, +, \cdot, ^*, 0, 1\rangle$ is a Boolean algebra, and C a binary relation on B which satisfies for all $x, y, z \in B$

C_0. $x(-C)0$.
C_1. If $x \neq 0$, then xCx.
C_2. If xCy, then yCx. (Symmetry)
C_3. If xCy and $y \leq z$, then xCz. (Monotonicity)
C_4. If $xC(y+z)$, then xCy or xCz. (Distributivity)

It is well known that the contact algebra of regular closed sets of a Euclidean space satisfies additionally

C_{ext}. If $\{z : xCz\} = \{z : yCz\}$, then $x = y$. (Extensionality)
C_{con}. If $x \neq 0$ and $x \neq 1$, then xCx^*. (Connectivity)

If C is a contact relation on B satisfying C_{ext} and C_{con}, then B needs to be atomless, and it was unknown whether such a relation exists on every atomless Boolean algebra. We can now show

Theorem 1. *On every atomless Boolean algebra there is a contact relation satisfying C_{ext} and C_{con}.*

This implies a representation theorem for atomless Boolean algebras:

Corollary 1. *For every atomless Boolean algebra B there are a compact connected topological space X and an isomorphism from B onto a dense subalgebra of the regular closed algebra of X.*

References

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